

# MONOTONIC LOCAL DECAY ESTIMATES

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ABSTRACT. For the Hamiltonian operator  $H = -\Delta + V(x)$  of the Schrödinger Equation with a repulsive potential, the problem of local decay is considered. It is analyzed by a direct method, based on a new,  $L^2$  bounded, propagation observable. The resulting decay estimate, is in certain cases **monotonic** in time, with no “Quantum Corrections”. This method is then applied to some examples in one and higher dimensions. In particular the case of the Wave Equation on a Schwarzschild manifold is redone: Local decay, stronger than the known ones are proved (minimal loss of angular derivatives and lower order of radial derivatives of initial data). The method developed here can be an alternative in some cases to the Morawetz type estimates, with  $L^2$ -multipliers replacing the first order operators. It provides an alternative to Mourre’s method, by including thresholds and high energies.

## SECTION 1

### 1. Introduction.

The starting point to a-priori estimates for dispersive equations is finding an operator which generates a monotonic function relative to the flow; the prime examples are Morawetz identity, the Dilation identity and the pseudo conformal identity. The Morawetz identity applies in three or more dimensions.

The above identities are generated by differential operators  $M$ , and we have

$$(1.1) \quad \frac{d}{dt} \langle \psi(t), M\psi(t) \rangle \geq 0$$

where  $\psi(t)$  is the solution of Schrödinger Equation at time  $t$ , the  $\langle, \rangle$  stands for the usual  $L^2$ -scalar product.

To derive the Morawetz estimate we choose ( $n$ -dimension)

$$(1.2) \quad M = -i \frac{\partial}{\partial r} - \frac{n-2}{r}, n \geq 3, \quad r \equiv |x|.$$

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The Dilation identity:  $M = \frac{1}{2}(x \cdot p + p \cdot x)$

$$(1.3) \quad p = -i\nabla_x$$

The Conformal identity

$$(1.4) \quad -M = \left| \frac{x}{t} - p \right|^2 t^\alpha + t^{2-\alpha} V(x) \quad , 0 \leq \alpha \leq 2.$$

The aim of this note is to construct **monotonic** observables  $M$  which are microlocal or phase-space operators.

The implications of such a construction include new local decay estimates, in particular, in one dimension, and new propagation estimates; it opens the way to new classes of a-priori estimates, including local decay at thresholds.

The operators  $M$  which I refer to, for obvious reason, as **propagation observables** (PROB) , are also known as **multipliers**.

## SECTION 2

### 2a. Some notation and preliminaries.

We consider the Schrödinger flow on  $L^2(\mathbb{R}^n)$  generated by a self-adjoint operator  $H$ :

$$(2.1) \quad i \frac{\partial \psi}{\partial t} = H\psi; \quad \psi(t=0) \in L^2.$$

We will focus on the case where

$$(2.2) \quad \begin{aligned} H &= -\Delta + V(x) \\ -\Delta &\quad \text{is the Laplacian on } \mathbb{R}^n. \end{aligned}$$

We assume from now on that  $V(x)$  is a real valued, uniformly bounded  $C^1$  function of  $x \in \mathbb{R}^n$ , so that  $H$  is self-adjoint on the domain  $D(-\Delta) = H^2(\mathbb{R}^n)$ , the Sobolev space.

In  $L^2(\mathbb{R}^n)$ , we define the momentum operator  $p$ ,

$$p = -i\partial_x \text{ and } r = |x|.$$

Then,

$$(2.4) \quad -\Delta = p \cdot p \equiv p^2.$$

We let

$$(2.5) \quad A = \frac{1}{2}(x \cdot p + p \cdot x) = x \cdot p - ni/2 = p \cdot x + ni/2 = \left( -ir \frac{\partial}{\partial r} - i \frac{\partial}{\partial r} r \right) / 2$$

and we have that

$$(2.6) \quad \imath[p_i, x_j] = \delta_{ij}; \quad i, j = 1, \dots, n,$$

where  $\delta_{ij}$  stands for the Kronecker delta function. Therefore:

$$(2.7) \quad \begin{aligned} \imath[-\Delta, x_j] &= 2p_j; & \imath[A, x_j] &= x_j \\ \imath[A, p_j] &= -p_j; & \imath[-\Delta, A] &= 2p^2. \end{aligned}$$

We denote  $\langle x \rangle^2 = 1 + |x|^2$  and by  $F(B \in I)$  the smoothed projection of the self-adjoint operator  $B$  in the interval  $I$ . E.g.,

$F(|x| \leq 1)$  stands for the multiplier

by the smoothed characteristic function of  $I \equiv \{|\lambda| \leq 1\}$  in  $L^2(\mathbb{R}_x^n)$ .

From equation (2.7) we derive ,

$$e^{\imath s A} p_j e^{-\imath s A} = e^{-s} p_j; \quad e^{\imath s A} x_j e^{-\imath s A} = e^s x_j.$$

**2b- Monotonic propagation Estimates.** It is generally known, from the works of Enss and Mourre that scattering states propagate into becoming "outgoing". So, in particular, one can prove, using the Mourre estimate that

$$\|P^-(A)e^{-iHt}g(H)P_c(H)\psi\| \leq o(t),$$

as  $t$  approaches  $+\infty$ . Here,  $P^-(A)$  is the projection on the negative spectral part of the Dilation generator,  $A$ . When  $g(H) \equiv 1$ , we get decay with essentially no rate. When  $g(H)$  is supported away from zero and infinity, one can prove fast decay in time, for localized initial data in space, as well as minimal and maximal velocity bounds [HSS and cited ref]. It is much more difficult to get estimates when the cutoff  $g$  is not present, and no localization of the initial data is assumed. In this case the methods of Mourre and [HSS] do not apply, in general. Some generalizations were obtained in [Ger, MRT and cited ref.], see also [Rod-T], replacing the Mourre estimate with a weak version of it. Here, I will develop a new way of getting decay estimates, for certain classes of hamiltonians, without localizing  $H$  or  $\psi$ . Furthermore, I will show that the propagation from the region of incoming waves into outgoing waves, and similar propagation estimates, is monotonic in time, for the free flow, and for the free flow perturbed by a class of repulsive potentials. These are two typical results: I show that it is possible to modify, by exponentially small corrections at infinity, the projection  $P^-(A)$  so that, the solution decays monotonically on its range, for the repulsive potentials:

$$\langle \psi(t), F_M^-(A)\psi(t) \rangle \downarrow 0, \text{ as } t \rightarrow +\infty$$

and

$$\int_0^T \|\langle x \rangle^{-1} F(A \leq -M) \psi(t)\|^2 dt \leq \langle \psi(0), 2F_M^-(A) \psi(0) \rangle,$$

see Proposition (6.2). The first part shows that the flow from incoming waves to outgoing is monotonic, with no restriction on the initial data! The second estimate shows, that at least locally in space, the incoming part is controlled, integrably in time, by the size of the incoming waves part of the initial data. So, in particular, no incoming wave can reappear locally, including zero energy and high energy contributions. The above estimates hold in any dimension, including one dimension, for one hump potentials. This has immediate applications to the case of scattering of the wave equation on Black-hole metrics:

**Theorem.** *For the Hamiltonian with the Schwarzschild potential  $\ell^2 V(x)$ , with  $V$  analytic repulsive, we have the following estimate:*

$$\int_0^T \|F(|x| \leq r_0) \ell u\|^2 \leq c \ln \ell E(u).$$

See section 8. Previously, a similar estimate was obtained in [B-Sof3,4], by complicated multi-step phase space propagation estimates. The propagation estimates above extends to time dependent hamitonians, with small, sufficiently localized potential perturbations.

### SECTION 3

#### 3. The propagation observable.

Since  $A$ , the dilation generator defined in equation (2.5), is a self-adjoint operator, we can construct the operator  $F(A/R)$ :

$$(3.1) \quad F\left(\frac{A}{R}\right) \equiv \tanh \frac{A}{R}$$

by the spectral theorem.

We show that  $F(A/R)$  has a positive commutator with  $H = -\Delta$ , and find lower bounds for it, if  $R$  is sufficiently large.

Then, this is extended to  $H = -\Delta + V$  for certain classes of potentials  $V$ .

Note that the analysis works in any dimension, and we specify to one dimension, which is the more difficult case.

To proceed, recall the commutator expansion Lemma [Sig-Sof1-2].

Let

$$ad_A^n(B) \equiv [ad_A^{n-1}(B), A]; \quad ad_A^1 = [B, A].$$

**Lemma 3.1.** *Commutator Expansion Lemma*

$$(3.2) \quad \begin{aligned} i[B, f(A)] &= \int \hat{f}(\lambda) e^{i\lambda A} [e^{-i\lambda A} B e^{i\lambda A} - B] d\lambda \\ &= f'(A) i[B, A] + \frac{1}{2!} f''(A) i[[B, A], A] + \cdots R_n \end{aligned}$$

$$(3.3) \quad R_n = \frac{1}{n!} \int \hat{f}(\lambda) e^{i\lambda A} \int_0^\lambda e^{-isA} \int_0^s e^{-i\mu A} \cdots \int_0^t e^{-iuA} (-i)^n \text{ad}_B^n(A) e^{+iuA} du \cdots d\lambda.$$

In particular, we get:

**Corollary 3.2.**

*Let  $A$  be the dilation generator as defined before, on  $L^2(\mathbb{R}^n)$ .*

*For  $R > 2/\pi$*

$$\tanh A/R : D(-\Delta) \rightarrow D(-\Delta).$$

*Proof.* Commuting  $\Delta$  through  $e^{i\lambda A/R}$ , we have:

$$e^{i\lambda A/R} [\Delta, e^{-i\lambda A/R}] = e^{i\lambda A/R} \Delta e^{-i\lambda A/R} - \Delta = (e^{-2\lambda/R} - 1) \Delta : D(\Delta) \rightarrow L^2.$$

Therefore, using the Commutator Expansion Lemma with  $n = 1$ , and the property (3.6) of the Fourier Transform of the tanh function, the result follows.

**Theorem 3.3.**  $i[-\Delta, \tanh(A/R)] = 2pg^2(A/R)p \geq 0$ , for  $R > 2/\pi$ . Here,

$$g^2(A/R) = \frac{\sin(2/R)}{\cosh \frac{2A}{R} + 2 \cosh \frac{2}{R}}.$$

*Proof.*

In the sense of forms, on  $D(H) \times D(H)$  :

$$\begin{aligned}
i[p^2, \tanh(A/R)] &= ip[p, \tanh(A/R)] + i[p, \tanh(A/R)]p \\
&= p \int \hat{f}(\lambda) e^{i\lambda A/R} (-i) \int_0^\lambda e^{-isA/R} (p/R) e^{isA/R} ds d\lambda + c.c. \\
&= p \int \hat{f}(\lambda) e^{i\lambda A/R} (-i) \int_0^\lambda e^{+s/R} (p/R) ds d\lambda + c.c. \\
(3.4) \quad &= -ip \int \hat{f}(\lambda) e^{i\lambda A/R} p e^{+s/R} \Big|_0^\lambda d\lambda + c.c. \\
&= -ip \int \hat{f}(\lambda) \left( e^{\lambda/R} - 1 \right) e^{i\lambda A/R} p d\lambda + c.c. \\
&= -ip \left[ \tanh \left( \frac{A+i}{R} \right) - \tanh \left( \frac{A}{R} \right) \right] p + c.c. \\
&= p \frac{1}{i} \left[ \tanh \frac{A+i}{R} - \tanh \frac{A-i}{R} \right] p
\end{aligned}$$

provided  $|\hat{f}(\lambda)| \leq ce^{-k|\lambda|}$  with  $k > \frac{1}{R}$ ,  $|\lambda| > 1$ .

We also note that

$$(3.5) \quad \hat{f}(\lambda) \left( e^{\lambda/R} - 1 \right) \sim \frac{1}{\lambda} \left( e^{\lambda/R} - 1 \right) \text{ near zero,}$$

which is bounded.

$$(3.6) \quad \hat{f}(\lambda) = \frac{\pi}{\sinh \pi \lambda}, \lambda > 0,$$

and similar formula for  $\lambda < 0$ .

$$\begin{aligned}
(3.7) \quad \frac{1}{i} \left( \tanh \frac{A+i}{R} - \tanh \frac{A-i}{R} \right) &= \frac{1}{i} \frac{\sinh(2i/R)}{\cosh \frac{A+i}{R} \cosh \frac{A-i}{R}} \\
&= \frac{\sin(2/R)}{\cosh \frac{2A}{R} + 2 \cosh \frac{2}{R}} > 0 \quad \text{for } R > 2/\pi.
\end{aligned}$$

□

**Corollary 3.4.** *Propagation estimate*

For  $R > 2/\pi$ ,  $H = -\Delta$

$$\begin{aligned}
(3.8) \quad &\left\langle \psi(t), \tanh \frac{A}{R} \psi(t) \right\rangle - \left\langle \psi(0), \tanh \frac{A}{R} \psi(0) \right\rangle \\
&= \int_0^t ds \|g(A)p\psi(s)\|^2 \leq 2\|\psi\|_{L^2}^2.
\end{aligned}$$

with  $g^2(A) \geq \frac{C}{R} \frac{1}{\cosh \frac{2A}{R}}$ .

*proof.*

For  $\psi \in D(H)$  :

$$\begin{aligned} & \frac{d}{dt} \left\langle e^{-iHt} \psi, \tanh \frac{A}{R} e^{-iHt} \psi \right\rangle \\ &= \left\langle H \psi(t), \tanh \frac{A}{R} \psi(t) \right\rangle - \left\langle \psi(t), \tanh \frac{A}{R} H \psi(t) \right\rangle \\ &= \left\langle \psi(t), \left( H \tanh \frac{A}{R} - \tanh \frac{A}{R} H \right) \psi(t) \right\rangle. \end{aligned}$$

The first equality follows by Von Neumann's Theorem. The second equality follows by Corollary 3.2 and Spectral Theorem. The Corollary now follows from Theorem 3.3 and Fundamental Theorem of Calculus.

Few remarks are in order now.

**Remark 1.** The above estimate shows that in the region  $|A| \leq C$ , the solution has an extra gain of **one** derivative, upon time averaging. One expects, more generally, that away from the propagation set, in the phase-space, that the gain in derivatives should be high.

Another important conclusion is the monotonicity of the flow in the phase space.

**Remark 2.** The corollary implies that the left hand side is monotonically increasing in time, in fact, with non vanishing derivative.

This means that the flow from the region  $A \leq 0$  to the region  $A \geq 0$  is strictly monotonic. This has important applications:

Define

$$(3.9) \quad F_M^+(A/R) \equiv \left( F \left( \frac{A - M}{R} \right) + 1 \right) / 2.$$

Then, the function  $F_M^+(A/R)$  is exponentially close to the projection operator  $P^+(A \geq M)$ . ( for  $|A|$  large enough depending on  $R$  ), the projection on outgoing waves.

We can then immediately conclude that outgoing part of the solution is strictly monotonic increasing up to exponentially small correction of order  $e^{-M}$ .

Moreover, since the solution decays in time in the complement region, we see that the correction is  $o(t)e^{-M}$ .

This property will remain true under decaying potential perturbations, in some sense, since for large  $M$ , the potential term is  $O(M^{-\sigma})$  in this region.

## SECTION 4

**4. Adding Potentials.**

The main interest in this note will be the case of “one hump” potentials in one dimension. These are repulsive potentials  $V$ , such that

$$(4.1) \quad i[V, A] = -x \cdot \nabla V \geq 0.$$

We begin with the simple model

$$(4.2) \quad V_0(x) = \frac{c_0}{b^2 + x^2}, \quad c_0 > 0.$$

Then, we have that Monotonic propagation estimates hold for  $H_0 = -\Delta + V_0(x)$ :

**Proposition 4.1.** *For  $H_0 = -\Delta + V_0(x)$ , as above,*

$$(4.3) \quad i[H_0, F(A/R)] = 2pg^2(A/R)p + c_0 \frac{2}{b^2 + x^2} x g^2(A/R) x \frac{1}{b^2 + x^2}$$

$$(4.4) \quad \begin{aligned} & 2 \int_0^t \|g(A/R)p\psi(s)\|^2 ds + c_0 \int_0^t \|g(A/R) \frac{x}{b^2 + x^2} \psi(s)\|^2 ds \\ &= \langle \psi(t), \tanh(A/R)\psi(t) \rangle - \langle \psi(0), \tanh(A/R)\psi(0) \rangle \end{aligned}$$

and  $g^2(A/R) \gtrsim \frac{C}{R} \cosh^{-1}(2A/R)$ , as before.

*Proof.* The proof follows from Theorem 3.3 and its application with  $x$  replacing  $p$ :

$$(\text{Thm 3.3}) : \quad i[-\Delta, F(A/R)] = 2pg^2(A/R)p$$

$$\begin{aligned} i[V_0, F(A/R)] &= +c_0 \frac{2}{b^2 + x^2} i[F(A), x^2] \frac{1}{b^2 + x^2} \\ &= c_0 \frac{2}{b^2 + x^2} x g^2(A/R) x \frac{1}{b^2 + x^2} \end{aligned}$$

where we use that  $i[F(A), x^2] = 2xg^2(A)x$ , the sign reversed when  $x \leftrightarrow p$ . Equation (4.4) follows upon integrating over time the Heisenberg identity for the Schrödinger equation.  $\square$

The above theorem, and its proof, extends in a variety of situations:



**Corollary 4.2.** *Let*

$$H = -\Delta + V(x),$$

*and suppose that  $V(x)$  admits a representation of the form:*

$$V(x) = \int_0^\infty \frac{\rho(\alpha)d\alpha}{\alpha + x^2}, \quad \rho(\alpha) \geq 0$$

*$\rho(\alpha)$  a positive measure,  $|\rho(\alpha)| \leq c|\alpha|$ ,  $|\alpha| \leq 1$ . We assume, moreover, that*

$$\int_0^\infty \frac{\rho(t)}{1+t} dt < \infty.$$

*Then, the estimates of Theorem 4.1 hold for  $H$ , with a different weight function in  $x$ :*

$$\frac{x}{b^2 + x^2} \rightarrow W_\rho(x)$$

*so that*

$$(4.5) \quad 2 \int_0^t \|g(A/R)p\psi(s)\|^2 ds + c \int_0^t \|g(A/R)W_\rho(x)\psi(s)\|^2 ds \\ \leq |\langle \psi(t), \tanh(A/R)\psi(t) \rangle| + |\langle \psi(0), \tanh(A/R)\psi(0) \rangle|$$

**Remark** The class of potentials  $V(x)$  above are Stieltjes functions.

*Proof.* The contribution from the potential term  $V$  to the commutator is computed as before, to be

$$(4.6) \quad \int_0^t ds \int_0^\infty \rho(\alpha) \|g(A/R) \frac{x}{\alpha + x^2} \psi(s)\|^2 d\alpha$$

$$(4.7) \quad g(A/R) \frac{x}{\alpha + x^2} \psi = g \left( 1 - \frac{\alpha + x^2}{\alpha_0 + x^2} \right) g^{-1} g \frac{x}{\alpha + x^2} \psi + g \frac{x}{\alpha_0 + x^2} \psi$$

Now, if we integrate over  $|\alpha - \alpha_0| < \delta|\alpha_0|$ ,  $\delta \ll 1$ , we have that

$$(4.8) \quad (\alpha_0 > 0), \int_{|\alpha - \alpha_0| < \delta\alpha_0} \|g(A/R) \frac{x}{\alpha + x^2} \psi\|^2 \rho(\alpha) d\alpha \geq \\ \int_{|\alpha - \alpha_0| < \delta\alpha_0} \|g(A/R) \frac{x}{\alpha + x^2} \psi\|^2 \rho(\alpha) d\alpha \\ - c \int_{|\alpha - \alpha_0| < \delta\alpha_0} \|g \frac{\alpha - \alpha_0}{\alpha_0 + x^2} g^{-1}\|^2 \|g \frac{x}{x^2 + \alpha^2} \psi\|^2 d\alpha.$$

So, we only need to get smallness of

$$\sup_{|\alpha - \alpha_0| < \delta \alpha_0} \left\| g \frac{\alpha - \alpha_0}{\alpha + x^2} g^{-1} \right\| \leq \sup_{|\alpha - \alpha_0| < \delta \alpha_0} 2\delta \left\| \frac{\alpha_0}{\alpha_0 + e^{2i\beta} x^2} \right\|_{L_x^\infty}$$

since in our case  $g^{-1} \sim \frac{e^{-A/R} + e^{A/R}}{2}$  and so  $\beta \sim 1/R$ .

So, for  $R > 1$ , the result follows. Summing over the intervals  $\alpha$  around  $\alpha_0 = \frac{k}{N}$ , for some large  $N$ ,  $0 < k$  integer, we get a lower bound on the expression (4.6) of the form

$$\begin{aligned} (4.9) \quad & \int_0^t ds C \sum_k \int_{|\alpha_k - \frac{k}{N}| < \delta k/N} \rho(\alpha) \left\| g(A/R) \frac{x}{\alpha_k + x^2} \psi(s) \right\|^2 d\alpha \\ &= C \int_0^t \sum_k \left\| g(A/R) \frac{\rho_k x}{\alpha_k + x^2} \psi(s) \right\|^2 ds \\ &\geq C \int_0^t ds \left\| g(A/R) \sum_k \frac{\rho_k \langle k \rangle^{1/2+\varepsilon} x}{\alpha_k + x^2} \psi \right\|^2 \equiv C \int_0^t \left\| g(A/R) W_\rho(x) \psi(s) \right\|^2 ds. \end{aligned}$$

$$\rho_k = \int_{|\alpha_k - \frac{k}{N}| < \delta k/N} \rho(\alpha) d\alpha$$

Next we need a **microlocal uncertainty principle** inequality:

**Lemma 4.3.** *For all  $R$  large enough,  $g$  a bounded  $C^\infty$  function,  $g(A/R) > 0$ , with,*

$$\sum_{i=1}^N |g^{(i)}| \leq c|g|,$$

for sufficiently large  $N = N(\sigma) > 2$ , we have:

$$(4.10) \quad (a) \quad (1 + \varepsilon) p g^2(A/R) p \geq g p^2 g$$

$$(b) \quad p^2 + \langle x \rangle_b^{-\sigma} x^2 \langle x \rangle_b^{-\sigma} \geq$$

$$(4.11) \quad \frac{1}{2} \langle x \rangle_b^{-\sigma} (p^2 + x^2) \langle x \rangle_b^{-\sigma} \geq \frac{1}{4} \langle x \rangle_b^{-2\sigma}$$

$$(c) \quad p g^2(A/R) p + \langle x \rangle_b^{-\sigma} x g^2(A/R) x \langle x \rangle_b^{-\sigma} \geq \frac{1-\varepsilon}{4} g(A/R) \langle x \rangle_b^{-2\sigma} g(A/R)$$

for all  $b$  large enough.

*Proof.* Part c) is proved using parts a) b). Assuming part b), we prove a) and c):

$$\begin{aligned}
 (4.12) \quad & g = g(A/R); \quad p = -i\nabla_x \\
 & pg^2p = pgp + [p, g]gp = gp^2g + gp[g, p] + [p, g]gp \\
 & = gp^2g + [gp, [g, p]] = gp^2g + g[p, p\tilde{g}] + [g, p\tilde{g}^*]p \\
 & = gp^2g + gp\tilde{g}^{(2)*}p - p\tilde{g}\tilde{g}^*p.
 \end{aligned}$$

$$\tilde{g} \equiv [g, p], \quad \tilde{g}^{(2)} \equiv [p, \tilde{g}].$$

So, since by construction  $\tilde{g} = O(\frac{1}{R})$ ,  $\tilde{g}^{(2)} = O(\frac{1}{R^2})$ , we get

$$\begin{aligned}
 p(g^2 + \tilde{g}\tilde{g}^*)p &= gp^2g + gp\tilde{g}^{(2)*}p \\
 &= gp^2g + pg\tilde{g}^{(2)*}p + p\tilde{g}^*\tilde{g}^{(2)*}p
 \end{aligned}$$

so,

$$(4.13) \quad p(g^2 + \tilde{g}\tilde{g}^* - 2\text{Reg}\tilde{g}^{(2)*})p = gp^2g.$$

Finally, for  $R$  large,

$$(4.14) \quad p(g^2 + \tilde{g}\tilde{g}^* - 2\text{Reg}\tilde{g}^{(2)*})p \leq (1 + \varepsilon_R)pg^2p$$

since  $g > 0$ , vanishing only at infinity, and since  $\tilde{g}, \tilde{g}^{(2)}$  decay faster at  $\infty$ , and are of order  $\frac{1}{R}$  and  $\frac{1}{R^2}$  respectively. We therefore conclude that part a) follows:

$$(4.15) \quad (1 + \varepsilon_R)pg^2p \geq gp^2g.$$

Next, we prove part c):

*Proof of c.* It follows from (4.15) that,

$$\begin{aligned}
 (4.16) \quad & pg^2(A/R)p + \langle x \rangle_b^{-\sigma} xg^2(A/R)x\langle x \rangle_b^{-\sigma} \\
 & \geq (1 - \varepsilon)g(A/R)p^2g(A/R) + (1 - \varepsilon)\langle x \rangle_b^{-\sigma} g(A/R)x^2g(A/R)\langle x \rangle_b^{-\sigma}.
 \end{aligned}$$

We now need to commute  $\langle x \rangle_b^{-\sigma}$  through  $g(A/R)$ . Commuting powers of  $\langle x \rangle_b^{-1}$  through, the error commutators terms are of the form

$$\left( \psi, \langle x \rangle_b^{-\sigma} \left( \frac{x}{\langle x \rangle_b} \right)^j \tilde{g}_1 a(x) x^2 \tilde{g}_2 \langle x \rangle_b^{-\sigma} \left( \frac{x}{\langle x \rangle_b} \right)^{j'} \psi \right), |a(x)| \leq c.$$

Any such term is therefore bounded by

$$c \left\{ \|x\tilde{g}_1 \left( \frac{x}{\langle x \rangle_b} \right)^j \langle x \rangle_b^{-\sigma} \psi\|^2 + \|x\tilde{g}_2 \left( \frac{x}{\langle x \rangle_b} \right)^{j'} \langle x \rangle_b^{-\sigma} \psi\|^2 \right\}.$$

For all  $f(x)$ , we have:

$$\begin{aligned} \|x\tilde{g}_1 f(x)\psi\| &\leq \|\tilde{g}_1 x f(x)\psi\| + \|\tilde{\tilde{g}}_1 x f(x)\psi\| \\ &\leq O\left(\frac{1}{R}\right) \|g x f(x)\psi\|, \end{aligned}$$

since  $\tilde{g}_1 = O\left(\frac{1}{R}\right)$ ,  $\tilde{\tilde{g}}_1 = O\left(\frac{1}{R^2}\right)$ , and  $|g'| + |g''| \leq c|g|$ .

Applying this last inequality with  $f(x) = \left(\frac{x}{\langle x \rangle_b}\right)^j \langle x \rangle_b^{-\sigma}$  we have that

$$\begin{aligned} \|x\tilde{g}_1 \left( \frac{x}{\langle x \rangle_b} \right)^j \langle x \rangle_b^{-\sigma} \psi\| &\leq O\left(\frac{1}{R}\right) \|g x \left( \frac{x}{\langle x \rangle_b} \right)^j \langle x \rangle_b^{-\sigma} \psi\| \\ &= O\left(\frac{1}{R}\right) \|g (\langle x \rangle_b^{-1} x)^j g^{-1} g x \langle x \rangle_b^{-\sigma} \psi\| \\ &\leq O\left(\frac{1}{R}\right) \|g (x \langle x \rangle_b^{-1})^j g^{-1}\| \|g x \langle x \rangle_b^{-\sigma} \psi\| \\ &\leq O\left(\frac{1}{R}\right) \|(e^{i\beta} x \langle e^{i\beta} x \rangle_b^{-1})\|_{L_x^\infty} \|g x \langle x \rangle_b^{-\sigma} \psi\|. \end{aligned}$$

So, for  $\beta$  sufficiently small ( $R > 1$ ), the error terms from commuting  $\langle x \rangle_b^{-\sigma}$  are smaller than

$$O\left(\frac{1}{R^2}\right) \|g x \langle x \rangle_b^{-\sigma} \psi\|^2.$$

Therefore, (4.16) implies

$$\begin{aligned} p g^2(A/R) p + \langle x \rangle_b^{-\sigma} x g^2(A/R) x \langle x \rangle_b^{-\sigma} &\geq \\ &\geq (1 - \varepsilon) g(A/R) p^2 g(A/R) + (1 - \varepsilon) g(A/R) \langle x \rangle_b^{-\sigma} x^2 \langle x \rangle_b^{-\sigma} g(A/R) \\ &\geq \frac{1 - \varepsilon}{4} g(A/R) \langle x \rangle_b^{-2\sigma} g(A/R) \end{aligned}$$

where the last inequality follows from part b).

*Proof of b).*

$$\begin{aligned}
p^2 + x^2 \langle x \rangle_b^{-2\sigma} &= \\
&= \langle x \rangle_b^{-\sigma} p^2 \langle x \rangle_b^{-\sigma} + (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) p^2 \langle x \rangle_b^{-\sigma} + \langle x \rangle_b^{-2\sigma} x^2 \\
&+ (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) p^2 (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) + \langle x \rangle_b^{-\sigma} p^2 (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) \\
&= \langle x \rangle_b^{-\sigma} p^2 \langle x \rangle_b^{-\sigma} + (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) p^2 (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) \\
&+ 2\sqrt{(b^{-\sigma} - \langle x \rangle_b^{-\sigma}) \langle x \rangle_b^{-\sigma/2}} p^2 \langle x \rangle_b^{-\sigma/2} \sqrt{(b^{-\sigma} - \langle x \rangle_b^{-\sigma})} \\
&+ 0 \left( \langle x \rangle_b^{-2\sigma-2} \left( \frac{x}{\langle x \rangle_b} \right)^2 \right) + x^2 \langle x \rangle_b^{-2\sigma}
\end{aligned}$$

which follows by commuting  $\langle x \rangle_b^{-\sigma/2}$  and  $\sqrt{(b^{-\sigma} - \langle x \rangle_b^{-\sigma})}$  through  $p^2$ .

For  $b \gg 1$ , the results follows:

$$\begin{aligned}
p^2 + x^2 \langle x \rangle_b^{-2\sigma} &\geq \langle x \rangle_b^{-\sigma} (x^2 + p^2) \langle x \rangle_b^{-\sigma} / 2 + x^2 \left( \frac{1}{2} \langle x \rangle_b^{-2\sigma} - c \langle x \rangle_b^{-\sigma-4} \right) \geq \\
&\geq \frac{1}{2} \langle x \rangle_b^{-\sigma} (x^2 + p^2) \langle x \rangle_b^{-\sigma} \geq \frac{1}{4} \langle x \rangle_b^{-2\sigma}.
\end{aligned}$$

□

#### Theorem 4.4.

Let  $V(x)$  be dilation analytic for all  $|s| \leq \beta$ . Then

$$i[V, \tanh A/R] = \frac{+i}{2 \cosh A/R} \left\{ V^{[-\beta]} - V^{[+\beta]} \right\} \frac{1}{\cosh A/R}$$

where

$$V^{[\beta]} \equiv e^{\beta A} V e^{-\beta A} = V(e^{-i\beta} x).$$

*Proof.*

$$\begin{aligned}
& i \left[ V, \frac{\sinh A/R}{\cosh A/R} \right] = \frac{1}{\cosh A/R} i[V, \sinh A/R] - \frac{1}{\cosh A/R} [V, \cosh A/R] \frac{\sinh A/R}{\cosh A/R} \\
& = \frac{1}{\cosh A/R} \{ i[V, \sinh A/R] \cosh A/R - i[V, \cosh A/R] \sinh A/R \} \frac{1}{\cosh A/R}. \\
& \{\cdots\} = \frac{i}{4} \left[ [V, e^\beta] - [V, e^{-\beta}] (e^\beta + e^{-\beta}) - ([V, e^\beta] + [V, e^{-\beta}]) (e^\beta + e^{-\beta}) \right] \\
& = [2[V, e^\beta] - 2[V e^{-\beta}] e^\beta] \frac{i}{4} \\
& = [2(V e^\beta - e^{+\beta} V) e^{-\beta} - 2(V e^{-\beta} - e^{-\beta} V) e^\beta] \frac{i}{4} \\
& = \frac{i}{2} [V - e^\beta V e^{-\beta} - V + e^{-\beta} V e^\beta] \\
& = \frac{i}{2} [V^{[-\beta]} - V^{[\beta]}].
\end{aligned}$$

□

## SECTION 5

### 5. Repulsive potentials and small Perturbations.

Let

$$(5.1) \quad H = -\Delta + V(x) + \varepsilon W(x)$$

where  $V, W$  as before, and have some analytic structure:

#### **Assumption AN**

For some  $\beta_0$  small, and  $|\beta| \leq \beta_0$

$$V(e^{\pm i\beta} x), W(e^{\pm i\beta} x)$$

are bounded, continuously differentiable, and decay at  $\infty$ ;

$$V, x \cdot \nabla V, (x \cdot \nabla)^2 V, W, x \cdot \nabla W, (x \cdot \nabla)^2 W$$

are all uniformly bounded by  $C\langle x \rangle^{-2}$ , and the same holds for the analytic continuations (with  $|\beta| \leq \beta_0$ ) above.

**Proposition 5.1.** *Let  $H$  as above, with  $V, W$  satisfying Assumption AN.*

*Then*

$$\begin{aligned}
 (5.2) \quad & i[H, \tanh A/R] = \\
 & = pg^2(A/R)p + \frac{1}{\cosh(A/R)}(i/2) [V(e^{i\beta}x) - V(e^{-i\beta}x)] \frac{1}{\cosh(A/R)} \\
 & + \varepsilon \frac{1}{\cosh(A/R)}(i/2) [W(e^{i\beta}x) - W(e^{-i\beta}x)] \frac{1}{\cosh(A/R)} \\
 & \beta = \frac{1}{R}.
 \end{aligned}$$

**Definition**

$V$  is analytic repulsive potential if

$$i(V(e^{i\beta}x) - V(e^{-i\beta}x)) \geq 0.$$

**Example**

$$V(x) = \frac{1}{1+x^2}.$$

In this case

$$\begin{aligned}
 i(V(e^{i\beta}x) - V(e^{-i\beta}x)) &= -2 \operatorname{Im} \frac{1}{1+e^{2i\beta}x^2} \\
 &= \frac{2x^2 \sin 2\beta}{|1+e^{2i\beta}x^2|^2} \geq c_\beta \frac{x^2}{\langle x \rangle^4}, c_\beta > 0
 \end{aligned}$$

provided  $|\beta| < \pi/4$ .

We conclude that

**Theorem 5.2.**

*Let  $H$  be as in (5.1), and  $V, W$  satisfy the assumption AN.*

*Suppose, moreover that  $V(x)$  is an analytic-repulsive potential, with lower bound*

$$i([V(e^{i\beta}x) - V(e^{-i\beta}x)]) \geq cx^2 \langle x \rangle^{-\sigma}, c > 0, \quad \sigma \geq 4,$$

*and  $W$  with decay of the above expression (to at least) of order  $\langle x \rangle^{-\sigma+2}$ . Then, for all  $\varepsilon$  small enough the RHS of equation 5.2 is positive and the corresponding local propagation estimates hold:*

$$\int_0^T \langle [\|g(A/R)p\psi(t)\|^2 + \|\langle x \rangle^{-1}g(A/R)\psi(t)\|^2 + \|x\langle x \rangle^{-2}g(A/R)\psi(t)\|^2] \rangle dt$$

$$\leq c|\langle\psi(T), (\tanh A/R)\psi(T)\rangle| + |\langle\psi(0), \tanh(A/R)\psi(0)\rangle|.$$

Here,

$$g^2(A/R) \sim \frac{1}{R} \frac{1}{\cosh^2 A/R}.$$

*Proof.* The only thing to check is that the  $W$  term in the commutator, is bounded by the repulsive contribution, coming from  $-\Delta + V$ . To this end, note that near  $x = 0$ ,

$$W(e^{i\beta}x) - W(e^{-i\beta}x) = x \int_{-\beta}^{\beta} e^{is} W(e^{is}x) ds$$

is  $\sim x$ .

□

### Remark

The condition of analyticity is technical, and is due to the fact that the propagation observable we use is exponentially localized, up to a constant, at  $\infty$ .

## SECTION 6

### 6. Local Decay and other propagation estimates.

The operator  $\tanh A/R$  can play the role leading to an analytic version of the projections on outgoing and incoming waves  $P^\pm(A)$ .

We define

$$(6.1) \quad F_M^+ = F \left( \frac{A - M}{R} \right) = \left( \tanh \frac{A - M}{R} + 1 \right) / 2$$

So,  $F_M^+$  is exponentially small (in  $M/R$ ) for  $A - M < 0$ .

Similarly, we define

$$(6.2) \quad F_M^- = F^- \left( \frac{A + M}{R} \right) = \left( 1 - \tanh \frac{A + M}{R} \right) / 2$$

We also notice the following inequality as a consequence of Thm 3.3, Lemma 4.3.a, and proposition 5.1:

### Theorem 6.1.

For  $H = -\Delta + V$  with  $V$  satisfying assumption  $AN$ , for all  $R$  large enough, we have that:

$$i[H, \tanh(A/R)] \geq (1 - \varepsilon)g_R(A)p^2g_R(A) + g_R(A)\tilde{V}_\beta g_R(A)$$



where

$$(6.3b) \quad g_R^2(A) \sim \frac{2}{R} \frac{1}{2 + \cosh 2A/R}$$

and

$$(6.3c) \quad 2\tilde{V}_\beta = iV(e^{i\beta}x) - iV(e^{-i\beta}x)$$

$$(6.3d) \quad V(e^{i\beta}x) = e^{-\beta A}V(x)e^{+\beta A}$$

$$(6.3e) \quad \beta = 1/R.$$

It is now easy to find classes of potentials for which we get monotonic decay estimates:

In one dimension we need either one of :

$$(i) \quad \tilde{V}_\beta \geq 0, \text{ or } 2p^2 \sin 2\beta + \tilde{V}_\beta \geq 0,$$

$$(ii) \quad \tilde{V}_\beta \geq x^2 \langle x \rangle_b^{-2\sigma} - \frac{1}{10} \langle x \rangle_b^{-2\sigma+2}, \quad \sigma \geq 2.$$

$$(iii) \quad V = V_1 + \varepsilon W$$

where  $V_1$  satisfies (ii) and  $\varepsilon \ll 1$ , and  $|W_\beta| \leq \langle x \rangle_b^{-2\sigma+2}$ .

(iv)

Suppose that  $-\Delta + V \geq 0$ .

Then,  $p^2 + \tilde{V}_\beta/(2 \sin 2\beta) = ap^2 + (1-a)(p^2 + V) + [\tilde{V}_\beta/(2 \sin 2\beta) - (1-a)V]$

so, we need  $\tilde{V}_\beta/(2 \sin 2\beta) - (1-a)V \geq 0$  for some  $0 < a \leq 1$ .

**In three dimensions** Monotonic Decay estimates hold whenever  $p^2 + \tilde{V}_\beta/(2 \sin 2\beta) \geq 0$ :

E.g., when,

$$(i) \quad \frac{1}{4|x|^2} + \tilde{V}_\beta/(2 \sin 2\beta) \geq 0, \text{ or when}$$

Suppose that  $H = -\Delta + V \geq 0$ .

Then, we require that

$$\begin{aligned}
 \text{(ii)} \quad p^2 + \tilde{V}_\beta/(2 \sin 2\beta) &= ap^2 + [(1-a)p^2 + \tilde{V}_\beta/(2 \sin 2\beta)] \\
 &= ap^2 + [-(1-a)V + \tilde{V}_\beta/(2 \sin 2\beta)] + (1-a)(p^2 + V) \\
 &\geq \frac{a}{4|x|^2} + [\tilde{V}_\beta/(2 \sin 2\beta) - (1-a)V] \geq 0.
 \end{aligned}$$

which may be useful when  $V$  has a negative part.

### Local Decay

We have that for  $F \equiv F(A/R) = \tanh A/R$

$$(6.5) \quad i[H, F] = 2pg^2(A)p + (1/\cosh(A/R))\tilde{V}_\beta(1/\cosh(A/R))$$

which we now assume to be positive:  $\tilde{V}_\beta \geq 0$ , and

$$\begin{aligned}
 i[H, F] &= 2pg^2(A)p + (1/\cosh(A/R))\tilde{V}_\beta(1/\cosh(A/R)) \\
 (6.6) \quad &\geq 2(1-\varepsilon)g(A)p^2g(A) + (1/\cosh(A/R))\tilde{V}_\beta(1/\cosh(A/R)) \geq \\
 &\geq g(A)B^2g(A), \quad \text{with } B^2 \geq 0.
 \end{aligned}$$

Occasionally we have

$$(6.7) \quad B^2 > \delta_{int}\chi(|x| \leq b_{int}) + \delta_{out}|\tilde{V}_\beta|$$

which is typical to one hump potentials  $V$ .

Now, let  $M$  be a large positive number, and recall the definition:

$$F_M^+(A/R) \equiv \left( \tanh \frac{A-M}{R} + 1 \right) / 2$$

and

$$F_M^-(A/R) = \left( 1 - \tanh \frac{A+M}{R} \right) / 2$$

the smooth projections on outgoing and incoming waves.

Then, letting for a momnet  $f(A-M) \equiv 1/\cosh \frac{A-M}{R}$ ,

$$\begin{aligned}
 \text{(6.8)} \quad i[H, 2F_M^+] &= 2pg^2(A-M)p + f(A-M)\tilde{V}_\beta f(A-M) \\
 &\geq 2(1-\varepsilon)g(A-M)p^2g(A-M) + f(A-M)\tilde{V}_\beta f(A-M)
 \end{aligned}$$

$$\begin{aligned}
 -i[H, 2F_M^-] &= 2pg^2(A+M)p + f(A+M)\tilde{V}_\beta f(A+M) \\
 \text{(6.9)} \quad &\geq 2(1-\varepsilon)g(A+M)p^2g(A+M) \\
 &+ f(A+M)\tilde{V}_\beta f(A+M).
 \end{aligned}$$

In particular, it follows, since  $-2F_M^- \leq 0$  that

**Proposition 6.2.**

$$(6.10a) \quad \langle \psi(t), F_M^- \psi(t) \rangle \downarrow 0, \text{ as } t \rightarrow +\infty$$

and

$$(6.10b) \quad \int_0^T \|Bg(A + M)\psi(t)\|^2 dt \leq \langle \psi(0), 2F_M^- \psi(0) \rangle,$$

with  $B$  is defined in equation (6.6).

This kind of monotonic decay is interesting, as it gives control of the solution in the classically forbidden regions in terms of the size of the solution at **time zero** with no corrections.

Applications will be discussed separately.

Next, we want to jack-up the decay estimate to a slowly decaying weight, rather than  $Bg$ .

For this, we introduce new propagation observables:

( $\sigma > 0$ )

$$(6.11) \quad \begin{aligned} 0 &\leq F_M^\pm (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) + (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) F_M^\pm \equiv F_M^\pm (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) + c.c. \\ \langle x \rangle_b^{-\sigma} &= (b^2 + |x|^2)^{-\sigma/2} \leq b^{-\sigma}. \end{aligned}$$

We then have:(c.c. stands for Hermitian conjugate)

**Proposition 6.3.**

$$(6.12) \quad \begin{aligned} i[H, F_M^+ (b^{-\sigma} - \langle x \rangle_b^{-\sigma}) + c.c.] &= \\ &= 2\sigma \langle x \rangle^{-\frac{\sigma}{2}-1} A(F_M^+) \langle x \rangle^{-\frac{\sigma}{2}-1} \\ &+ \sum_i \tilde{F}_M C_i \tilde{F}_M + O(R^{-a}) F_M O(1) A \tilde{F}_M \end{aligned}$$

where  $\tilde{F}_M$  stands for approximate (discrete) derivatives of  $F_M$  (w.r.t.  $A$ ), and  $C_i, O(1)$ , are operators which are of higher order in  $\langle x \rangle^{-1}$ , and of order  $R^{-1}$  at least,  $R$ -large.

*Proof.* We denote  $\langle x \rangle_b \equiv \langle x \rangle$ , and let  $g(A) \equiv 1/\cosh \frac{A-M}{R}$ .

$$(6.13) \quad \begin{aligned} i[H, F_M^+ (b^{-\sigma} - \langle x \rangle^{-\sigma}) + c.c.] &= \\ &= i[H, F_M^+] (b^{-\sigma} - \langle x \rangle^{-\sigma}) + (b^{-\sigma} - \langle x \rangle^{-\sigma}) i[H, F_M^+] \\ &+ F_M^+ i[p^2, -\langle x \rangle^{-\sigma}] + c.c. \\ &= g(A)(2 \sin 2\beta p^2 + \tilde{V}_\beta) g(A) (b^{-\sigma} - \langle x \rangle^{-\sigma}) + (b^{-\sigma} - \langle x \rangle^{-\sigma}) g(A)(2 \sin 2\beta p^2 + \tilde{V}_\beta) g(A) \\ &+ F_M^+ \sigma [\langle x \rangle^{-\sigma-1} x p + p x \langle x \rangle^{-\sigma-2}] + c.c. \equiv I + I^* + J. \end{aligned}$$

We symmetrize  $J$  first:

Since

$$\begin{aligned}
 A &= \frac{1}{2}(x \cdot p + p \cdot x) = x \cdot p - ni/2 = p \cdot x + ni/2, \\
 J &= \sigma F_M^+ [\langle x \rangle^{-\sigma-2} A + A \langle x \rangle^{-\sigma-2}] + c.c. \\
 (6.14) \quad &= \langle x \rangle^{-\sigma-2} \sigma A (F_M^+) + \sigma A (F_M^+) \langle x \rangle^{-\sigma-2} \\
 &= \sigma \left[ [AF_M^+, \langle x \rangle^{(-\sigma-2)/2}], \langle x \rangle^{(-\sigma-2)/2} \right] + 2\sigma \langle x \rangle^{(-\sigma-2)/2} AF_M^+ \langle x \rangle^{(-\sigma-2)/2}.
 \end{aligned}$$

We need to know that we can write

$$[F(A), \langle x \rangle^{-2}] \text{ as } \tilde{F}(A)C$$

with  $C$  bounded, of order  $\langle x \rangle^{-2}$ , at least.

Now,

$$\begin{aligned}
 (6.15) \quad [F(A), \langle x \rangle^{-2}] &= -\langle x \rangle^{-2} [F(A), x^2] \langle x \rangle^{-2} \\
 &= -\langle x \rangle^{-2} 2x \tilde{F}(A) x \langle x \rangle^{-2}.
 \end{aligned}$$

Then, using that  $\beta = \frac{1}{R}$  is small, we can write for any  $\beta'$ ,  $(g_{\beta'}(A) \equiv 1/\cosh(\beta' A))$

$$\begin{aligned}
 \langle x \rangle^{-2} x \tilde{F}(A) x \langle x \rangle^{-2} &= g_{\beta'}(A) \cosh(\beta' A) x \langle x \rangle^{-2} \tilde{F}(A) x \langle x \rangle^{-2} \cosh(\beta' A) g_{\beta'}(A) \\
 &= \frac{1}{2} g_{\beta'}(A) (x \langle x \rangle^{-2})_{\beta'} \cosh 2\beta' A \tilde{F}(A) (x \langle x \rangle^{-2})_{\beta'} g_{\beta'}(A) \\
 &+ \frac{1}{2} g_{\beta'}(A) (x \langle x \rangle^{-2})_{-\beta'} \cosh(-2\beta' A) \tilde{F}(A) (x \langle x \rangle^{-2})_{-\beta'} g_{\beta'}(A) \\
 &+ \frac{1}{2} g_{\beta'}(A) (x \langle x \rangle^{-2})_{-\beta'} \tilde{F}(A) (x \langle x \rangle^{-2})_{+\beta'} g_{\beta'}(A) \\
 &+ \frac{1}{2} g_{\beta'}(A) (x \langle x \rangle^{-2})_{+\beta'} \tilde{F}(A) (x \langle x \rangle^{-2})_{-\beta'} g_{\beta'}(A)
 \end{aligned}$$

where

$$\begin{aligned}
 (x \langle x \rangle^{-2})_{\beta}' &= e^{i\beta'} x \langle e^{i\beta'} x \rangle^{-2} = e^{i\beta'} x (1 + e^{2i\beta'} x^2)^{-1} \\
 &= e^{i\beta'} x (1 + e^{-2i\beta'} x^2) (1 + e^{2i\beta'} x^2)^{-1} (1 + e^{-2i\beta'} x^2)^{-1} \\
 2Re \left[ e^{i\beta'} x (1 + x^2 \cos 2\beta' - x^2 2i \sin 2\beta') (1 + x^4 + 2x^2 \cos 2\beta')^{-1} \right] &= O(\langle x \rangle^{-1})
 \end{aligned}$$

and similarly for the Imaginary part, (for  $\beta'$  small). Here we choose  $\beta' \leq \beta$ .

$$\tilde{F}(A) \sim O\left(\frac{1}{R}\right) (\cosh 2\beta A)^{-1},$$

$$g_{\beta'}(A) = (\cosh \beta' A)^{-1}.$$

So, we have that

$$(6.16) \quad [F(A), \langle x \rangle^{-2}] = g(A) C g(A).$$

Similarly, we can rewrite

$$\begin{aligned}
 [F(A), \langle x \rangle^{-2}] &= -2\langle x \rangle^{-2} \tilde{F}_-(A) x^2 \langle x \rangle^{-2} \\
 &= +2\langle x \rangle^{-2} \tilde{F}_-(A) \left( \frac{1}{1+x^2} - 1 \right) \\
 &= -2\langle x \rangle^{-2} \tilde{F}_-(A) + 2\langle x \rangle^{-2} \tilde{F}_-(A) \left( \frac{1}{1+x^2} \right) (\cosh \beta' A) (\cosh \beta' A)^{-1} \\
 &= -2\langle x \rangle^{-2} \tilde{F}_-(A) + 2\langle x \rangle^{-2} \left\{ \tilde{F}_-(A) e^{\beta' A} \left( \frac{1}{1+x^2} \right)_{\beta'} + \tilde{F}_-(A) e^{-\beta' A} \left( \frac{1}{1+x^2} \right)_{\beta'} \right\} (\cosh \beta' A)^{-1} \\
 (6.17) \quad &= \sum C_i \tilde{F}(A), \quad \tilde{F}(A) \sim (\cosh \beta' A)^{-1}
 \end{aligned}$$

$$C_i = O(\langle x \rangle^{-2} \frac{1}{R}), \quad \beta' \leq \beta, \text{ small.}$$

Using the above identities for  $[F(A), \langle x \rangle^{-2}]$  we can easily symmetrize the expressions for  $I, I^*$  and  $J$  to get:

$$\begin{aligned}
 J &= \langle x \rangle^{-\frac{\sigma}{2}-1} 2\sigma A(F_M^+) \langle x \rangle^{-\frac{\sigma}{2}-1} \\
 &+ [\langle x \rangle^{-\frac{\sigma}{2}-1}, [\langle x \rangle^{-\frac{\sigma}{2}-1}, \sigma A(F_M^+)]] \\
 (6.18) \quad &= \langle x \rangle^{-\sigma/2-1} 2\sigma A(F_M^+) \langle x \rangle^{-\frac{\sigma}{2}-1} \\
 &+ [\langle x \rangle^{-\frac{\sigma}{2}-1}, [\langle x \rangle^{-\frac{\sigma}{2}-1}, \sigma A(F_M^+)]]
 \end{aligned}$$

Using that for any  $Q$ ,

$$(6.19a) \quad [[Q, f(A)], g(A)] = [Q, g]f - f[Q, g] = [[Q, g], f]$$

$$\begin{aligned}
 (6.19b) \quad i[\langle x \rangle^{-m}, A F_M^+] &= +m\langle x \rangle^{-m-2} x^2 F_M^+ + A i[\langle x \rangle^{-m}, F_M^+] \\
 &= m\langle x \rangle^{-m-2} x^2 F_M^+ - A C \tilde{F}_M = m\langle x \rangle^{-m-2} x^2 F_M^+ - C A \tilde{F}_M - [A, C] \tilde{F}_M
 \end{aligned}$$

$$[A, C] = O(\langle x \rangle^{-m}/R).$$

Commuting again with  $(m \equiv \sigma/2 + 1) \langle x \rangle^{-\frac{\sigma}{2}-1}$  we get that the double commutator is of the form:

$$O(\langle x \rangle^{(-\sigma-2)/2}/R) \tilde{F}_M \cdot O(\langle x \rangle^{(-\sigma-2)/2}/R).$$

Therefore

$$\begin{aligned} J &= \langle x \rangle^{-\sigma/2-1} 2\sigma A(F_M^+) \langle x \rangle^{-\frac{\sigma}{2}-1} \\ &+ O(R^{-1})(\langle x \rangle^{-\frac{\sigma}{2}-1} \tilde{F}_M) O(1) F_M \langle x \rangle^{-\frac{\sigma}{2}-1} \\ (6.20) \quad &\geq \sigma \langle x \rangle^{-\frac{\sigma}{2}-1} |A| F_M^+ \langle x \rangle^{-\frac{\sigma}{2}-1} \end{aligned}$$

Symmetrizing  $I + I^*$ , we have that, as above:

$$\begin{aligned} I + I^* &= \tilde{F}_M H_\beta \tilde{F}_M \chi_b^2(|x|) + \chi_b^2(|x|) \tilde{F}_M H_\beta \tilde{F}_M \\ &= \tilde{F}_M 2\chi_b(|x|)(p^2 + \tilde{V}_\beta) \chi_b(|x|) \tilde{F}_M + \tilde{F}_M O(\langle x \rangle^{-\sigma-2}) \tilde{F}_M + \\ (6.21) \quad &\tilde{F}_M O(\langle x \rangle^{-\sigma} R^{-1}) H_\beta \tilde{F}_M + c.c. \\ &\geq \tilde{F}_M \chi_b(|x|)(2 \sin 2\beta p^2 + \tilde{V}_\beta) \chi_b(|x|) \tilde{F}_M; \end{aligned}$$

$$\chi_b(|x|) = (b^{-\sigma} - \langle x \rangle_b^{-\sigma})^{1/2}$$

Combining (6.20), (6.21) we have that:

**Theorem 6.4.** (*Local Decay for Analytic Repulsive Potentials*)

Let  $H = -\Delta + V(x)$  as before and s.t.  $V$  is Analytic repulsive, and  $-\Delta + V \geq 0$ .

Then, for  $\sigma > 0$ ,

$$\begin{aligned} &\int_0^T \|\langle A \rangle^{1/2} F_M^+ \langle x \rangle^{-\sigma-1} \psi\|^2 dt \\ &+ \int_0^T \|p \chi_b(|x|) \tilde{F}_M \psi\|^2 dt \leq C \|\psi\|^2 \end{aligned}$$

**Remark:**

We can replace  $A F_M^+$  by  $\langle A \rangle$  in the expression for  $J$ , eq (6.20), using the local decay estimate proposition (5.2), which controls the region  $|A| \leq C$ , and a similar bound on  $F_M^-$ .

Similar estimate holds for  $F_M^-$ :

**Theorem 6.5.** *(Pointwise (and integral) decay of Incoming waves)*

*Under the conditions of Theorem 6.4, we have that*

$$\int_0^T \|\langle A \rangle^{1/2} F_M^- \langle x \rangle^{-\sigma-1} \psi\|^2 dt + \langle \langle x \rangle^{-\sigma} \psi(T), (F_M^-)^A \langle x \rangle^{-\sigma} \psi(T) \rangle \leq 2 \langle \psi(0) (F_M^-)^2 \psi(0) \rangle.$$

Combining all the above, we get that local decay holds with the following weight:

$$\int_0^T \|\langle A \rangle^{1/2} \langle x \rangle^{-1-\varepsilon} \psi\|^2 dt \leq c \|\psi\|^2.$$

## SECTION 7

### 7. Applications: Schwarzschild manifolds, generalized repulsive potentials.

When the Hamiltonian  $H \geq 0$ , we can get the desired decay estimates by simply verifying that

$$2 \sin 2\beta p^2 + V_\beta > 0$$

for some  $\beta$  small.

In particular, if  $-x \cdot \nabla V > 0$ , together with some uniformity of the analytic continuations  $V_\beta$ , the above inequality follows.

We also get local decay, for one hump potentials, including the Schwarzschild for each fixed angular momentum:

Case Study: Schwarzschild potentials

Here we solve the wave equation

$$-\frac{\partial^2 u}{\partial t^2} = Hu$$

$$u_0 = (f_0, g_0) \in H^1 \otimes L^2$$

Let

$$(7.1) \quad H = -\partial_{r_*}^2 + V_\ell(r) \text{ on } L^2(\mathbb{R}, dr_*)$$

where  $r_* = r + 2M \ln(r - 2M)$

so that  $\frac{dr_*}{dr} = 1 + 2M \frac{1}{r-2M} = \frac{r-2M+2M}{r-2M}$

$$(7.2) \quad = \frac{r}{r-2M} \text{ and } \frac{dr}{dr_*} = \frac{r-2M}{r} = 1 - \frac{2M}{r}.$$

$$(7.3) \quad V_\ell(r) = \left(1 - \frac{2M}{r}\right) \frac{2M}{r^3} + \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} \ell(\ell+1) \quad \ell = 0, 1, 2, \dots$$

Since for each  $\ell$ ,  $V_\ell(r)$  is a one hump potential around the point

$$(x^2 = \ell(\ell+1)) \quad r_\ell \equiv \frac{3M(\lambda^2 - 1) + M\sqrt{\rho(\lambda^2 - 1)^2 + 32\lambda^2}}{2\lambda^2}$$

$$(7.4) \quad \alpha_\ell^* = \alpha^*(r = r_\ell); \alpha_\infty^* = r_*(r = 3M)$$

it follows that the decay estimates hold for analytic each  $H_\ell$ , if we can show that the humps are repulsive! Summing over all  $\ell$ , after multiplying by  $P_\ell$ , the projection on the  $\ell$ 'th spherical harmonic, local decay follows for  $-\Delta$  on Schwarzschild manifolds.

This argument applies to all manifolds where the resulting potential is one-hump, analytic repulsive at fixed angular momentum.

In fact we get somewhat different and new estimates in this case, since, as we remarked before, the propagation observable (PROB) we use is bounded on  $L^2$ , unlike the Morawetz estimate and its various generalizations which are bounded from  $H^{1/2} \rightarrow L^2$ . The solution of the wave equation can be written in terms of the initial data  $u(x, t=0) := f_0, \dot{u}(x, t=0) := g_0$  as:

$$u(x, t) := U(t)u_0 = \cos(\sqrt{H}t)f_0 + \frac{\sin \sqrt{H}t}{\sqrt{H}}g_0.$$

There is a fundamental new difficulty with the WE (Wave Equation) as compared with the Schrödinger equation. This is due to the fact that  $L^2$  norm can grow linearly in time for the WE, and the LHS of the propagation estimate (PRES) has a form different from the Schrödinger case.

**Theorem 7.1.** *Local Decay-WE*

$$(7.5) \quad \int_0^T \|\langle x \rangle^{-3/2} \langle A \rangle^{1/2} u\|^2 dt < CE^{1/2}(u_0) \left( E^{1/2}(g(H \leq \epsilon)u_0) + c_\epsilon \|u_0\|_{L^2} \right).$$

The proof of the Theorem is a consequence of the propositions that follow:

The Heisenberg equation formulation of the wave equation is

$$(7.6a) \quad \partial_t[(u, B\dot{u}) - (\dot{u}, Bu)] = [H, B]$$

where

$$(7.6b) \quad -\partial_t^2 u = Hu.$$



Using  $B \equiv \frac{\partial}{\partial t}$ , we get the **Energy Identity**:

$$\begin{aligned} & \frac{\partial}{\partial t} [(u, \frac{\partial}{\partial t} \dot{u}) - (\dot{u}, \frac{\partial}{\partial t} u)] \\ &= \frac{\partial}{\partial t} [(u, -Hu) - (\dot{u}, \dot{u})] \\ &= \frac{\partial}{\partial t} \int |\nabla u|^2 + |\dot{u}|^2 + V(x)|u|^2 dx \equiv \frac{\partial}{\partial t} E(u) = 0. \end{aligned}$$

So, the energy conservations reads  $E(u) = E(u_0)$ . In our case

$$(7.6c) \quad B = i \tanh A/R.$$

First, we reduce the problem to initial data with localized frequencies near zero.

For this, let  $g = g(|p| \leq 1)$ ,  $\bar{g} = 1 - g$  and write  $u$  as

$$\begin{aligned} u &= gu + \bar{g}u \\ (u, F\dot{u}) - (\dot{u}, Fu) &= (gu, Fg\dot{u}) + (\bar{g}u, Fg\dot{u}) \\ &+ (\bar{g}u, F\bar{g}\dot{u}) + (gu, F\bar{g}\dot{u}) - (g\dot{u}, Fgu) - (\bar{g}\dot{u}, Fgu) \\ &- (\bar{g}\dot{u}, F\bar{g}u) - (g\dot{u}, F\bar{g}u). \end{aligned}$$

Every term with  $\bar{g}u$  is good.

$$\bar{g}u = \bar{g}|p|^{-1} \langle p \rangle \langle p \rangle^{-1} pu$$

and therefore  $|(\psi, \bar{g}u)| \leq \|\psi\| \|\bar{g}|p|^{-1} \langle p \rangle\| \|\langle p \rangle^{-1} pu\|$ .

Next, we have

$$\begin{aligned} (gu, F\bar{g}\dot{u}) - (\bar{g}\dot{u}, Fgu) &= \\ &= (pgu, F(\frac{1}{p})\bar{g}\dot{u}) + (pgu, \tilde{F}_+ \frac{1}{p} \bar{g}\dot{u}) \\ &- (\frac{1}{p} \bar{g}\dot{u}, Fpgu) - (\frac{1}{p} \bar{g}\dot{u}, \tilde{F}_- pgu) \\ &\leq 2\|\frac{1}{p} \bar{g}\| \|\dot{u}\| (\|F\| + 2\|\tilde{F}\|) \|pgu\|. \end{aligned}$$

Finally to deal with terms with no  $\bar{g}$  in them, we need to exploit the fact that  $\tanh A/R$  vanishes linearly in  $A$  near zero.

$$\begin{aligned} & - (g\dot{u}, Fgu) + (gu, Fg\dot{u}) = \\ & - \left( g\dot{u}, F \frac{1}{A} (xp - i/2) gu \right) + \left( (xp - i/2) gu, \frac{1}{A} F g\dot{u} \right) \\ & = - \left( g\dot{u}, (F \frac{1}{A} x \tilde{g}) gpu \right) + (gpu, \tilde{g} x \frac{1}{A} F g\dot{u}), \quad \tilde{g}g = g. \end{aligned}$$

Now, since  $F = i \tanh A/R$ ,  $\pm iF = \mp G$  with  $G = G^*$ . Furthermore,  $g$  acts like the convolution with the function  $\hat{g}$ , the Fourier transform of  $g$ , which is real.

Hence  $gu, g\dot{u}$  are real.

This leads to the cancellation of the two terms with  $-\frac{i}{2}$  factor.

We are therefore left with

$$\begin{aligned} & |2(g\dot{u}, (FA^{-1}x\tilde{g})gpu) + 2(gpu, (\tilde{g}x\frac{1}{A}F)g\dot{u})| \\ & \leq 4\|g\dot{u}\|\|FA^{-1}x\tilde{g}gpu\|. \end{aligned}$$

Hence, collecting all the terms, we arrive at

$$\begin{aligned} & |\langle u, i \tanh A/R \dot{u} \rangle - \langle \dot{u}, i \tanh A/R u \rangle| \\ (7.7) \quad & \leq c\|\dot{u}\| \|\langle p \rangle^{-1} pu\| + c\|g\dot{u}\|\|FA^{-1}xgpu\|. \end{aligned}$$

To this end, we need the following propagation observables, and energy decomposition; Fix a (large) time  $T$ . We break the initial data  $(f, g) = u_0$  as:

$$\begin{aligned} u_0 &= F(H \leq T^{-1})u_0 + F(H \geq T^{-1})u_0 \\ &= u_l + u_h \equiv F_{<}u_0 + F_{>}u_0 \end{aligned}$$

Clearly then, since  $H$  commutes with the dynamics  $U(t)$ ,

that

$$\begin{aligned} U(t)u_l &= F(H \leq T^{-1})U(t)u_l \\ U(t)u_h &= F(H \geq T^{-1})U(t)u_h, \end{aligned}$$

so that:

$$\begin{aligned} \|HU(t)u_l\|_{L^2} &= \|HF_{<}U(t)u_l\| \\ &\leq T^{-\frac{1}{2}}\|H^{\frac{1}{2}}U(t)u_l\|_{L^2}E^{\frac{1}{2}}(u_l). \end{aligned}$$

We will use the following propagation observables:

$$\begin{aligned} B_1 &= i \tanh(A/R) \\ (7.8) \quad B_2 &= iF_M^\pm(A/R) \\ B_n^\sigma &\equiv F_M^\pm i\langle x \rangle^{-m} + c.c. \quad m \geq 0. \end{aligned}$$

We then have, as before, that

$$\begin{aligned} [H, B_1] &= \tilde{g}_0(A)H_\beta\tilde{g}_0(A) \\ (7.9) \quad [H, B_2] &= \pm\tilde{g}_M^\pm(A)H_\beta\tilde{g}_M^\pm(A) \\ [H, B_m^\sigma] &= F_M^\pm \{ \langle x \rangle^{-m-2}A + c.c. \} F_M^\pm \pm \{ \tilde{g}_M^\pm H_\beta \tilde{g}_M^\pm \langle x \rangle^{-m} F_M^\pm + c.c. \} \end{aligned}$$

Next, we have the following preliminary estimates on the *LHS* of the Heisenberg identity:

**Lemma 7.2.**

(7.10)

- (i)  $|\langle \dot{u}_h, B_1 u_h \rangle - \langle u_h, B_1 \dot{u}_h \rangle| \leq C \|\dot{u}_h\|_{L^2} E^{\frac{1}{2}}(u_h) T^{\frac{1}{2}}$
- (ii)  $|\langle \dot{u}, B_1 u \rangle - \langle u, B_1 \dot{u} \rangle| \leq C \|\chi(|A| \leq \lambda) \dot{u}\|_{L^2} \|\chi(|A| \leq \lambda) (\tanh A/R) u\|_{L^2}$   
 $+ C \|\chi(|A| \geq \lambda) e^{-|A/R|} \dot{u}\|_{L^2} \|\chi(|A| \geq \lambda) e^{-|A/R|} u\|_{L^2}$
- (iii) *Similar bounds hold for  $B_1$ , with  $A \rightarrow A - M$ .*
- (iv)  $|\langle \dot{u}, B_m^\sigma u \rangle - \langle u, B_m^\sigma \dot{u} \rangle| \leq C E(u_0) \quad \text{for } m \geq 1.$

*Proof.*

- (i) Follows by Cauchy-Schwarz inequality and

$$\|u_h\|_{L^2} = \|H^{-\frac{1}{2}} F(H \geq \frac{1}{T}) H^{\frac{1}{2}} u_h\| \leq T^{\frac{1}{2}} E^{\frac{1}{2}}(u_h).$$

- (ii) Follows by noting that on support  $\chi_{\geq}(A)$  :

$$|(\tanh(A/R) \mp 1) \chi_{\geq}(A)| \lesssim 2e^{(-2|A/R|)} \chi_{\geq}(A)$$

and that

$$\langle \dot{u}, Bu \rangle - \langle u, B\dot{u} \rangle \equiv \langle B \rangle_u^{Heis} = 0$$

for  $B = 1$  ( or any reality preserving symmetric operator).

- (iii) Follows from (i), (ii) by replacing  $A$  by  $A - R$ .

- (iv) Follows from  $\|\langle x \rangle^{-1} u\|_{L^2} \leq C E^{\frac{1}{2}}(u)$ , and that

$$[\langle x \rangle^{-1}, F_M^\pm] = -\langle x \rangle^{-1} [\langle x \rangle, F_M^\pm] \langle x \rangle^{-1} \simeq \langle x \rangle^{-1} \langle x \rangle \tilde{F}_M^\pm \langle x \rangle^{-1},$$

with  $\tilde{F}_M^\pm$  - bounded.

We proceed to estimating  $\langle B_1 \rangle_{u_h}^{Heis}$ .

**Proposition 7.3.**

There exists a sequence of times,  $T_n \rightarrow \infty$ , such that

$$(7.11) \quad \|F(|x| \leq MT_n^{\frac{1}{2}}) \tilde{g}(A) u_h(T_n)\|_{L^2} \leq CT_n^{\frac{1}{4}} M.$$

*Proof.*

Applying the previous propagation estimates with  $B_1$ , and using Lemma (7.2)(i), it follows that :

$$\int_0^T \|\tilde{g}(A) \langle x \rangle^{-1} u_h\|^2 dt + \int_0^T \|\tilde{g}(A) p u_h(t)\|^2 dt \leq CT^{\frac{1}{2}} E(u_h).$$

Next, we apply the cutoff in  $|x|$  :

$$\|F(|x| \leq M\sqrt{T}) \tilde{g}(A) u_h(t)\| \leq M\sqrt{T} \|\langle x \rangle^{-1} \tilde{g}(A) u_h(t)\|,$$

so that

$$\begin{aligned} \int_0^T \|F(|x| \leq MT^{\frac{1}{2}}) \tilde{g}(A) u_h(t)\|^2 dt \\ \leq M^2 T \int_0^T \|\langle x \rangle^{-1} \tilde{g}(A) u_h(t)\|^2 dt \leq CM^2 T^{\frac{3}{2}}. \end{aligned}$$

Therefore,  $\exists T_n \rightarrow \infty$  s.t.

$$\|F(|x| \leq M\sqrt{T_n}) \tilde{g}(A) u_h(T_n)\|^2 \leq CM^2 T_n^{\frac{3}{2}} T_n^{-1},$$

since, otherwise

$$\int_0^{T_n} \|F(|x| \leq M\sqrt{T}) \tilde{g}(A) u_h(t)\|^2 dt > CM^2 T_n^{\frac{3}{2}}.$$

Next, we use the above proposition to bound  $\langle B_1 \rangle_{u_h}^{Heis}$ , using the fact that if  $|x| > M\sqrt{T}$ , and  $|p| \geq \frac{1}{\sqrt{T}}$ , then, classically, in the phase space,  $A \gtrsim M$ , which, together with the localization in  $A$ , via  $\tilde{g}(A)$ , gives fast decay in  $M$ , for  $|x| > M\sqrt{T}$ .

**Proposition 7.4.**

$$|\langle B_1 \rangle_{u_h}^{Heis}| \leq C \|F(|x| \leq MT^{\frac{1}{2}}) \tilde{g}(A) u_h(t)\| \|\dot{u}\|_{L^2} \\ + O(M^{-\infty}) TE(u).$$

*Proof.*

We need to bound

$$\|F(|x| > MT^{\frac{1}{2}}) \tilde{g}^2(A) F(H \geq T^{-1}) u_h(t)\|_{L^2} \\ \leq TE^{\frac{1}{2}}(u_h) \|F(|x| > MT^{\frac{1}{2}}) \tilde{g}^2(A) F(H \geq T^{-1})\|_{L^2 \rightarrow L^2}$$

where we used that

$$\|u_h(t)\| \leq CtE^{\frac{1}{2}}(u_h).$$

To this end, we write the above operator product as

$$\|F(|x| > MT^{\frac{1}{2}}) x^{-2n} x^{2n} \tilde{g}^2(A) H^n H^{-n} F(H \geq \frac{1}{T})\| \\ \leq \|x^{2n} H^n \tilde{g}^2(A)\| M^{-2n} T^{-n} \\ + \|x^{2n} [\tilde{g}^2(A), H^n]\| M^{-2n}$$

The first term on the *RHS* has a factor  $x^{2n} H^n$ , which, when expanded, is a sum of terms of the form  $x^{2n} P^{2j} V^k \dots$  and such that the order in  $x$  is at most  $2n - 2j$  and the order in  $p$  is  $2n - 2j$  in each monomial  $P_j \sim x^{2n} V^k p^j \dots$

This is because our  $V(x)$  decays at least like  $|x|^{-2}$ . Hence, we can always pair each monomial to be

$$P_j \sim x^{2n-2j} p^{2n-2j} \left(1 + O\left(\frac{1}{x}\right)\right) C_j.$$

Each such monomial can be rewritten as

$$P_j \sim C'_j \left(1 + O\left(\frac{1}{|x|}\right) + O(A^{-1})\right) A^{2n-2j}$$

Hence the first term on the *RHS* is bounded by

$$\sum_j C''_j \|A^{2n-2j} \tilde{g}_1(A)\|.$$

The second term on the *RHS* is similar:

$$x^{2n}[\tilde{g}^2(A), H^n] \sim \sum_j x^{2n}[\tilde{g}^2(A), P_j] \sim \sum_k g_{k'}(A)x^{2k}p^{2k} \sim \sum_l g_l(l)A^{2l}.$$

Using the exponential bound on the  $\tilde{g}^2(A)$ , and noting that the number of terms is at most of order  $n^n$ , we get a bound of the form ( after inserting  $x^N H^{N/2}$ )

$$M^{-N} \sum_{n=1}^N n^n A^n e^{-|A/R|} \lesssim \left( \frac{N^2 R}{2eM} \right)^N.$$

If we choose  $N^2 \sim R^{-1}\sqrt{M}$ , we get a bound  $\sim M^{-N/2} \sim M^{-\frac{1}{2R}}M^{\frac{1}{4}}$ .  $\square$

The propositions above imply ( after choosing  $M \gtrsim C \ln T$ ).

**Theorem 7.5.**

$$\left| \langle B_1 \rangle_{u_h}^{Heis}(T_n) \right| \leq C T_n^{\frac{1}{4}} \ln T_n E(u_h).$$

The above process, beginning with the bound of Lemma (7.2)(i), is now iterated (  $\ln T$  times...), where we use the above  $T^{\frac{1}{4}}$  bound to replace the  $T^{\frac{1}{2}}$  bound of Lemma(i). This will give a  $T^{\frac{1}{8}}$  bound etc...

We conclude that

**Theorem(7.6).**

$$\langle B_1(T_n) \rangle_{u_h}^{Heis} \leq C E(u_h), \quad T_n \rightarrow \infty.$$

Next, we need to bound  $\langle B_1 \rangle_{u_l}^{Heis}$ .

The method is similar to the previous case; however, the propagation observables used need to be iterated, and the argument is a bit more involved.

To this end we consider the part of the data where  $H \leq T^{-1}$ , that is, estimating  $u_l$ .

In this case the propagation observable we use is of the general form

$$F_M^\pm i\langle x \rangle^{-\sigma} F_M^\pm = B_{\sigma,M}^\pm. \quad \sigma \geq 1.$$

The commutator with  $H$  has two parts; one comes from  $\langle x \rangle^{-\sigma}$  and another from  $F_M^\pm$ . They have apposite sign, and therefore, we need to control one of them in terms of the other.

Since  $\sigma \geq 1$ , the *LHS* of the Heisenberg equation is uniformly bounded in time:

$$\langle B_{\sigma,M}^\pm \rangle_{u_l}^{Heis} \leq CE(u_l).$$

We have that

$$\begin{aligned} [H, B_{\sigma,M}^\pm] &= -\sigma F_M^\pm \langle x \rangle^{-\sigma-2} A F_M^\pm + c.c. \\ &\quad \pm [g_M^\pm H_\beta g_M^\pm \langle x \rangle^{-\sigma} F_M^\pm + c.c.] \equiv C_\sigma + D_\sigma. \end{aligned}$$

Our goal is to show that in some sense  $D_\sigma$  is higher order, so the  $C_\sigma$  term will give a propagation estimate. We iterate on  $\sigma$  to get the final bound.

Symmetrizing  $C_\sigma$ , as before, we get

$$\begin{aligned} C_\sigma &= -\sigma \langle x \rangle^{(-\sigma-2)/2} A F_M^\pm \langle x \rangle^{(-\sigma-2)/2} + O(\langle x \rangle^{-\sigma-2} \tilde{g}(A)/R) \\ &= -\sigma \langle x \rangle^{(-\sigma-2)/2} A F_M^\pm \langle x \rangle^{(-\sigma-2)/2} + O(\langle x \rangle^{-\sigma-2} \tilde{g}(A)/R) \end{aligned}$$

$$\begin{aligned} |\langle u_l, D_\sigma u_l \rangle| &\leq C \|\tilde{g}_M(A) \langle x \rangle^{-\sigma} u_l\| \|H_\beta \tilde{g}_M(A) F(H \leq 1/T) u_l\| \\ &\leq C \{ \|\tilde{g}_M H_\beta F_{<} u_l\| + \|[p^2, \tilde{g}_M] F_{<} u_l\| + \|V_\beta \langle x \rangle^2\| \|\langle x \rangle^{-2}, \tilde{g}_M\| F_{<} u_l\| \} \|\tilde{g}_M \langle x \rangle^{-\sigma} u_l\| \\ &\leq C \left[ \frac{T^{-1/2}}{\sqrt{R}} \|H_\beta^{1/2} u_l\| + \|g_{1,M}(A) p^2 F_{<} u_l\| \right] \|\tilde{g}_M \langle x \rangle^{-\sigma} u_l\| \\ &\quad + C \frac{1}{\sqrt{R}} \|\langle x \rangle^{-2} F_{<} u_l\| \|\tilde{g}_M \langle x \rangle^{-\sigma} u_l\| \\ &\leq \frac{C}{\sqrt{R}} T^{-1/2} E^{1/2}(u_l) \|\tilde{g}_M \langle x \rangle^{-\sigma} u_l\| + \frac{C}{\sqrt{R}} \|\tilde{g}_M \langle x \rangle^{-\sigma} u_l\| O(L^2(dt)). \end{aligned}$$

using that

$$\begin{aligned} \langle x \rangle^{-2} &\leq C H_\beta \\ p^2 &\leq C H_\beta. \end{aligned}$$

$$\begin{aligned}
\|\tilde{g}_M(A)\langle x \rangle^{-\sigma} u_l\|^2 &= \langle \tilde{g}_M(A)\langle x \rangle^{-\sigma} u_l, \tilde{g}_M(A)\langle x \rangle^{-\sigma} u_l \rangle \\
&\leq \frac{C}{R} |\langle u_l, C_\sigma u_l \rangle| + C e^{-M/2R} \|\tilde{g}(A)\langle x \rangle^{-\sigma} u_l\|^2
\end{aligned}$$

Provided

$$\sigma \geq (\sigma + 2)/2$$

$$\tilde{g}(A) \sim (\cosh(A/R))^{-1}.$$

Putting it all together, we have:

**Proposition 7.7.**

For  $\sigma \geq (\sigma + 2)/2$ ,

$$\begin{aligned}
|\langle u_l, [H, B_{\sigma, M}^\pm] u_l \rangle| &\geq |\langle u_l, C_\sigma u_l \rangle| \\
&\quad - \frac{C}{M^{1/2}R} T^{-1/2} E^{1/2}(u_l) |\langle u_l, C_\sigma u_l \rangle|^{1/2} \\
&\quad - C e^{-M/2R} \|\tilde{g}(A)\langle x \rangle^{-\sigma} u_l\| T^{-1/2} E^{1/2}(u_l)
\end{aligned}$$

For  $B_{\sigma, M}^-$ ,  $\langle u_l, C_\sigma u_l \rangle$  is positive, and  $\langle u_l, C_\sigma u_l \rangle$  is negative for  $B_{\sigma, M}^+$ .

We integrate over time the Heisenberg equation and using the above proposition to obtain the following propagation estimate:

**Proposition 7.8.**

For  $\sigma \geq \frac{\sigma}{2} + 1$  :

$$\begin{aligned}
&\int_0^T \|\langle A \rangle^{1/2} \tilde{F}_M^\pm \langle x \rangle^{(-\sigma-2)/2} u_l\|^2 dt \\
&\leq C T^{-1/2} E^{1/2}(u_l) \int_0^T \|\tilde{F}_M^\pm \langle x \rangle^{(-\sigma-2)/2} u_l\| ds \\
&\quad + C T^{-1/2} e^{-M/2R} \int_0^T \|\tilde{g}(A)\langle x \rangle^{-\sigma} u_l\| ds + C E(u_l)
\end{aligned}$$

where the last term comes from  $\langle B_\sigma \rangle_{u_l}^{Heis}$ .

Applying the above result with  $\sigma = 2$ , we can get the following local decay estimate:



**Theorem 7.9.**

*Under the previous assumptions on the Hamiltonian, including the case of Schwarzschild potential, we have that*

$$\int_0^T \|\langle A \rangle^{1/2} F_M^\pm \langle x \rangle^{-2} u\|^2 dt \leq CE(u).$$

*Proof.*

The proof for the  $u_l$  part is completed by the above theorem, on noticing that for  $\sigma = 2$ , we have that

$$\int_0^T \|\tilde{g}(A) \langle x \rangle^{-2} u_l\| ds \leq C \int_0^T T^{-1/2} E^{1/2}(u_l) dt \leq CT^{1/2} E(u_l).$$

**Analytic Repulsiveness of the Schwarzschild potentials.**

When the potential vanishes at - infinity, exponentially fast, the situation is complicated by the fact that, even though

$$-x \cdot \nabla V \geq 0 \text{ at infinity,}$$

in general,  $V_\beta$  is not positive, but oscillates no matter how small  $\beta$  is:

For  $V(x) = e^{-x}$  for  $x \gg M$ ,

$$V_\beta = 2Im e^{-e^{-i\beta}x} = 2Im e^{-x \cos \beta} e^{+xi \sin \beta}$$

$= 2e^{-x \cos \beta} (+\sin(x \sin \beta))$  which decays exponentially, but oscillates with period  $(\sin \beta)^{-1}$ .

So, to prove analytic repulsiveness, we need to show that

$$2 \sin 2\beta p^2 + V_\beta \sim 2 \sin 2\beta p^2 + 2e^{-x \cos \beta} \sin(x \sin \beta)$$

is a positive operator.

**Theorem 7.10.**

*Suppose  $V(x)$  is repulsive:  $-x \frac{\partial V}{\partial x} \geq f^2(x) > 0$ , one hump potential, with non degenerate maximum.*

*Suppose, moreover, that  $V_\beta$  exists and is analytic for all  $|\beta|$  sufficiently small, and*

(i)  $|V_\beta(x)| \leq Ce^{-\delta x}, x > x_0, \text{ for some } x_0 > 0.$

$$(ii) \quad |V_\beta(x)| \leq C\langle x \rangle^{-2-a} \text{ for all } |x| > +x_0, \text{ some } a > 0.$$

condition (ii) can be replaced by condition iii):

$$(iii) \quad V_\beta(x) \geq f^2(x, \beta) > 0 \text{ for } x < -x_0.$$

Then,  $V$  is analytic-repulsive, and

$$(7.7) \quad 2 \sin 2\beta p^2 + V_\beta \geq \delta_0 \langle x \rangle^{-2}.$$

### Remarks

The condition on  $V$  implicitly implies that  $V$  has a (dilation) analytic extension from  $\mathbb{R}$  to the domain

$$\{e^{i\beta'} x \mid |\beta'| \leq \beta, x - \text{real}\}.$$

*Proof.*

Using the fundamental theorem of calculus and Taylor series expansion, we write  $V_\beta$  as

$$(7.8) \quad \begin{aligned} V_\beta(x) &= \int_{-\beta}^{\beta} 2 \{ \text{Im} \frac{\partial}{\partial s} V(e^{-is} x) \} ds \\ &= -2xV'(x)\beta + 2 \int_{-\beta}^{\beta} \text{Re} \{ e^{-is'} - is' V''(e^{-is'} x) \} s x^2 ds \\ &\geq -2xV'(x)\beta - \beta^2 \frac{|x|^2}{2} \sup_{|s'| \leq \beta} |V''(e^{-s'} x)|. \end{aligned}$$

Using the following Cauchy estimates

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \sup_{|z-z_0|=R} |f(z)|,$$

we have that:

$$V_\beta \geq -2xV'(x)\beta - \beta^2 |x|^2 2 \sup_{s'} \sup_{|z-e^{is'} x|=1} |V(z)|$$

where  $-\beta \leq s' \leq \beta$ .

Using condition (i) of the theorem, it follows that for large  $x$  positive,  $x > x_0$ :

$$(7.9) \quad V_\beta(x) \geq -2xV'(x)\beta - c\beta^2 |x|^2 e^{-\delta x}.$$

For  $x < x_0$ :

Since  $V(x)$  is assumed to be a one hump potential,  $xV'(x)$  is strictly positive away from zero, and (non degenerate case)

$$xV'(x) \sim \frac{1}{2}a^2x^2 \text{ near zero.}$$

Here  $x = 0$  is the top of the hump of  $V(x)$ . Since  $|V(z)| \leq C\langle x \rangle^{-2}$ , choosing  $\beta$  sufficiently small, we have that for all  $|x| \leq x_0$ .

$$(7.10) \quad -2xV'(x)\beta - 2\beta^2|x|^2 \sup_{|s'| \leq \beta} \sup_{|z - e^{is'}x|=1} |V(z)| \geq -xV'(x)\beta.$$

For  $x$  large, negative, we use the Cauchy estimate with  $|z - e^{is'}x| = 1$  replaced by a circle, which encloses  $e^{is'}x$  of radius  $\sim \delta(\beta|x|)^{1-\eta}$ ;  $\eta, \delta$  small, so that  $z$  is in the domain of analyticity.

Then, we have that for  $x < -x_0$ :

$$(7.11) \quad \begin{aligned} V_\beta(x) &\geq -2xV'(x)\beta - c\beta^{2\eta}|x|^{2\eta}\langle x \rangle^{-2}\langle x \rangle^{-2-a}/\delta^2 \\ &\geq -2xV'(x)\beta - c\beta^{2a}\langle x \rangle^{-2-\varepsilon}/\delta^2 \end{aligned}$$

for  $2\eta < a$ :

If condition (iii) is satisfied then

$$V_\beta(x) > 0 \text{ for all } x < -x_0.$$

Now, since  $xV'(x) > 0$  for  $x \neq 0$ , by choosing  $\beta$  sufficiently small, we have that

$$(7.12) \quad V_\beta(x) \geq \frac{\beta}{2}f^2(x) \text{ for all } x < x_0$$

and

$$(7.13) \quad |V_\beta(x)| \leq Ce^{-\delta x}, x < x_0.$$

So, to complete the proof, we need to show that  $2\sin 2\beta p^2 + V_\beta(x) > 0$ .

To this and, we use the uncertainty principle, which, in one dimension, gives (in one of its forms... [BS*t*])

$$(7.14) \quad p^2 + \lambda\chi_I \geq \frac{C(\lambda, I)}{1 + |x|^2}$$

where  $\chi_I$  is the characteristic function of the interval  $I$  :

By 7.9 - 7.14,

$$\begin{aligned}
2 \sin 2\beta p^2 + V_\beta(x) &\geq 2 \sin 2\beta \left( p^2 + \frac{1}{2} f^2(x) \chi(x < x_0) \right) - C \chi(x > x_0) e^{-\delta x} \beta^2 x^2 \\
&\geq 2 \sin \beta \left( p^2 + \frac{1}{2} f^2(x) \chi(x < x_0) \right) - C \beta^2 \chi(x > x_0) e^{-\delta x} x^2 (1 + x^2) \frac{1}{1 + x^2} \\
&\geq \frac{1}{1 + x^2} \left[ \beta C(x_0, f^2) - \chi(x > x_0) C \beta^2 e^{-\delta x_0/2} e^{-\delta(x-x_0/2)} x^2 (1 + x^2) \right] \\
&\geq \frac{\beta}{2} C(x_0, f^2) (1 + x^2)^{-1},
\end{aligned}$$

by choosing  $\beta$  small, and by choosing  $x_0$  large enough so that  $x_0 \gg \frac{2}{\delta}$ , to get

$$C \beta^2 e^{-\frac{\delta x_0}{2}} \delta^{-4} < \frac{\beta}{2} C(x_0, f^2)$$

which is possible, since increasing  $x_0$  only increases the value of  $C(x_0, f)$ .

□

**Theorem 7.11.** (*Improved Local Decay for Schwarzschild potentials*)

Let

$$H = -\Delta + V_\ell(x) \quad x \in \mathbb{R}.$$

$V_\ell$  is defined in (7.1) – (7.3) ( $x \equiv r_*$ ).

Then, the following local decay estimate holds:

$$(i) \quad \ell \geq 1 : \int_0^T dt \|Ju(x, t)\|^2 \leq C E^{1/2}(u) E^{1/2}(\langle p \rangle^{-1} p u)$$

$$J = J(x), |J(x)| \leq (1 + x^2)^{-1/2-\delta}.$$

$$\ell = 0;$$

$$(ii) \quad \int_0^T \|Ju(x, t)\|^2 dt \leq C E^{1/2}(u) E^{1/2}(\langle p \rangle^{-1} p u)$$

with

$$J = J(x), |J(x)| \leq (1 + x^2)^{-3/2-\delta}.$$

*Proof.*

For each  $\ell$ , the potential  $V_\ell(x)$  is a one hump function [B-Sof1], and has analytic continuation [Bac-Bac, Zw] for all  $\beta$  sufficiently small.

Moreover, it satisfies the conditions of Theorem 7.2 [Bac-Bac, Zw], and  $-x \cdot \nabla V \geq f^2(x)$  is also known [B-Sof1].

So, applying Theorems 7.2, 6.4 and proposition 7.1 the result follows.  $\square$

**Example** *Negative Potentials in 3 dimensions*

$$V(x) = - \left( \frac{a}{b^2 + x^2} \right)^2 \text{ in three dimensions.}$$

Then

$$-(2 - \varepsilon)V - x \cdot \Delta V = \frac{-(2 - \varepsilon)(b^2 + r^2) + 4r^2}{b^2 + r^2} V = \frac{(2 + \varepsilon)r^2 - b^2(2 - \varepsilon)}{b^2 + r^2} V$$

since

$$-x \cdot \nabla V = \frac{+4r^2}{(b^2 + r^2)} V.$$

The above expression is negative for  $r^2 \geq b^2 \frac{2 - \varepsilon}{2 + \varepsilon}$ .

Hence

$$\begin{aligned} 2p^2 - x \cdot \nabla V &= (2 - \varepsilon)(p^2 + V) + [-x \cdot \nabla V - (2 - \varepsilon)V] + \varepsilon p^2 \\ &\geq (2 - \varepsilon)\delta|x|^{-2} + \frac{\varepsilon}{4}|x|^{-2} - \frac{(2 + \varepsilon)r^2 - b(2 - \varepsilon)}{b^2 + r^2} \frac{a^2}{(b^2 + r^2)^2} \\ &> 0, \text{ for } (a/b) \text{ sufficiently small.} \end{aligned}$$

**Example** *-Addition of Humps*

This example is typical to the problem of constructing a propagation observable with no  $\ell$  dependence for the Schwarzschild/Kerr problem, for example.

Here, I consider a simple example, leaving the general case to other works.

So, let

$$V(x) = \frac{2}{1 + |x|^2} + a \frac{1}{1 + |x - b|^3}$$

for  $a, b > 0, -\infty < x < \infty$ .

Then

$$-x \cdot \nabla V = \frac{4r^2}{(1 + r^2)^2} + \frac{3a|x - b|^3}{(1 + |x - b|^3)^2} + \frac{3ab \operatorname{sgn}(x - b)|x - b|^2}{(1 + |x - b|^3)^2}.$$

This expression may be negative for  $0 \leq x \leq b$ . It is negative near  $x = 0, x > 0$  since the last term dominates.

However, using the localized uncertainty principle, Lemma 4.3b, it follows that

$$\frac{1}{4}p^2 + \frac{4r^2}{(1+r^2)^2} \geq \frac{1}{4} \frac{1}{(1+r^2)^2}.$$

Therefore, we can easily arrange

$$2p^2 - x \cdot \nabla V \geq \varepsilon \langle x \rangle^{-4}$$

by choosing  $a$  small or  $b$  small.

### Other Perturbations

All the previous examples will still satisfy the local decay estimates under the addition of a small, fast decaying, possibly time dependent perturbation,  $W(x, t)$ , provided  $W_\beta(x, t)$  is well defined for small  $\beta$ , and satisfies the same size and decay conditions.

## SECTION 8

### 8. High Angular Momentum Bounds.

In this section we demonstrate an application to Schwarzschild scattering, for large angular momentum. It is by no means supposed to be comprehensive, and the genral results, including pointwise estimates will be developed elsewhere. In the previous sections we did not follow the dependence of the decay estimates on the  $\ell$  dependence. Here, we will consider the angular dependence of the previously obtained decay estimates, for the Schwarzschild potential and for the case where

$$V_\ell = \ell^2 V(x),$$

with  $V(x)$  analytic repulsive. This is motivated by the case of extreme Reissner Nordstrom Blackhole manifold. Our main goal is to show, that for large  $\ell$ , the local decay estimate holds, with a factor of  $\ell$ , up to log correction. Previously, this was proved in [B-Sof3,4 ], by a complicated generalized phase-space analysis. We begin with the following preliminary results, that follow directly from applying the previous estimates. First, we note that, in the Schwarzschild case, the behavior of the potential at large negative  $x_*$ , is  $\ell^2$  times an exponentially decaying function. Therefore, to insure that such a potential is repulsive analytic, we need to choose  $\beta$ , in the definition of the PROB, to be smaller than  $c(\ln \ell)^{-1}$ , for some sufficiently large positive  $c$ . Then, we have the following estimates:

**Proposition 8.1.** *Let*

$$(8.1) \quad H = -\Delta + \ell^2 V(x),$$

with  $V(x)$  analytic repulsive, for  $\beta \leq \beta_0(\ell)$ . Then, we have the following PRES:

$$(8.2) \quad \int_0^T \|H_\beta^{1/2} \tilde{g}(A/R)u\|^2 dt \leq cE(u)^{1/2} E^{1/2}(\langle p \rangle^{-2} pu),$$

$$(8.3) \quad \int_0^T \|Q_\beta^{1/2} \tilde{g}(A/R)\dot{u}\|^2 dt \leq cE(u),$$

where we define  $Q := \sqrt{H}$ .

$$(8.4) \quad \int_0^T \|J(x)\ell u\|^2 dt \leq cE(u),$$

where  $J(x) = cx \langle x \rangle^{-2} (1 + e^{bx})^{-1}$ , with  $b$  positive.

The proof of the above statements follows from application of the previous PRES to the hamiltonian defined in (8.1).

**Sketch of Proof** The estimate (8.2) follows by using the PROB  $\tanh(A/R)$ , together with Theorem (4.4).  $R$  is chosen large enough, depending on  $\ell$ , to insure the positivity of  $H_\beta = \frac{i}{2} [H^{[-\beta]} - H^{[\beta]}]$ . Here we note that the algebraic proof of Theorem (4.4) applies verbatim with  $H$  replacing  $V$ . The resulting PRES is the estimate (8.2).

The estimate (8.3) follows by repeating the above argument for the Schrödinger type equation, with hamiltonian given by  $Q = \sqrt{H}$ . To this end, we note that the equation satisfied by the function  $\dot{u}$ , is given by

$$\dot{u}(x, t) = \sin(Qt)(Qf) + \cos(Qt)g,$$

with  $Qf, g$  in  $L^2$ . The sine and cosine functions are linear combinations of  $e^{\pm iQt}$ , which is the propagator of the Schrödinger equation with hamiltonian  $\mp Q$ . Applying as above Theorem (4.4) and the resulting PRES, we obtain (8.3). To prove the estimate (8.4), we write the PRES for the following PROB

$$G := b(x)\partial_x + \partial_x b(x),$$

with  $b(x) = x/\langle x \rangle$ . Then, we obtain a positive term from the commutator with the potential part, of the form

$$-2b(x)\ell^2 V'(x),$$

together with two terms from the commutator with the Laplacian part of the hamiltonian. One term is positive, and is of second order in the radial derivative; the other is localized in  $x$ . This localized term, has coefficient of order 1, that is, independent of  $\ell$ . It comes from  $b'''$  term in the commutator. Since we proved that for such localized weight function the PRES holds, the result follows.

**Proposition 8.2.** *Under the same assumptions of Theorem (8.1), we have the following PRES: ( $r_0 > 0$ )*

$$(8.5) \quad \int_0^T \left\{ \|x F(|x| \leq r_0) \ell \tilde{g}(A/R) u\|^2 + \|F(|x| \leq r_0) \sqrt{\ell} \tilde{g}(A/R) u\|^2 \right\} \leq c [\langle F(A/R) \rangle_u^{\text{Heis}} - \langle F(A/R) \rangle_{u_0}^{\text{Heis}}].$$

The above proposition is a consequence of previous decay estimates, with  $V$  replaced by  $\ell V$ . The second term on the rhs, is bounded, with a loss  $\ell^{1/2}$ , to eliminate the vanishing  $x$  factor. This follows from application of the uncertainty principle, as in [DSS2].

We will be able to get the desired estimate from this last bound, on using it with  $u \rightarrow Q^{1/2}u$ , and using the fact that  $Q > F(|x| \leq r_0)\ell$ . The resulting estimate is restricted to the support of the operator  $g(A/R)$ . To this we show how to remove this projection from the estimate.

**Proposition 8.3.** *Under the same assumptions of Theorem (8.1), we have the following PRES: ( $r_0 > 0$ )*

$$(8.6) \quad \int_0^T \| \langle x \rangle^{-a} |A|^{1/2} F_M(A/R) u \|^2 dt \leq c \Re \left[ \left\langle \langle x \rangle^{-a'} F(A/R) \right\rangle_u^{\text{Heis}} - \left\langle \langle x \rangle^{-a'} F_M(A/R) \right\rangle_{u_0}^{\text{Heis}} \right],$$

where  $a = (a' + 1)/2$ .

This follows, as before, by using the following PROB, similar to the one used before, with similar computations:

$$\langle x \rangle^{-a'} F_M + F_M \langle x \rangle^{-a'}.$$

We can then use the PRES to remove the cutoff function  $g(A/R)$  from the PRES of proposition (8.2), except that we need to bound this error term by a quantity that is of order  $\ell^{-2}$ , up to possibly log corrections, for large  $\ell$ . The first power of  $\ell$  comes from , as before, by applying the above proposition to  $u \rightarrow Q^{1/2}u$ . To obtain another power of  $\ell$ , we use the **redeeming property** of the Heisenberg identity for the wave equation: If  $N$  is a symmetric reality preserving linear operator, then:

$$(8.7) \quad \langle N \rangle_u^{\text{Heis}} = 0.$$

In particular, this holds for  $N = 1, f(x), g(|p|), fg + gf$  with  $f, g$  real valued functions. Therefore, as noted before, we have

$$\begin{aligned} \langle F_M(A/R) \rangle_u^{\text{Heis}} &= \\ \langle F((A - M)/R \sim K_0) \rangle_u^{\text{Heis}} &+ \langle F(|(A - M)/R| \geq K_0) F_M \rangle_u^{\text{Heis}}. \end{aligned}$$



$$\begin{aligned}
\langle F(|(A-M)/R| \geq K_0) \rangle_u^{\text{Heis}} &= \langle (F_1(|x| \leq C) + \bar{F}_1(|x| \geq C)) F(|(A-M)/R| \geq K_0) \rangle_u^{\text{Heis}}, \\
\langle \bar{F}_1(|x| \geq C) F(|(A-M)/R| \geq K_0) \rangle_u^{\text{Heis}} \\
&= \langle F_x(F_p + \bar{F}_p) F(A) \rangle_u^{\text{Heis}} = \langle F_x F_p \rangle_u^{\text{Heis}} + \langle F_x \bar{F}_p \rangle_u^{\text{Heis}} + O(\ell^{-2}) = O(\ell^{-2}) + \langle F_x \bar{F}_p \rangle_u^{\text{Heis}}.
\end{aligned}$$

Here,  $F_x \equiv \bar{F}_1(|x| \geq C)$ ,  $F_p \equiv F_p(|p| \geq \delta\ell)$ ,  $\delta \geq 0$ .  $F(A) \equiv F(|(A-M)/R| \geq K_0)$ . We are therefore left with controlling (by  $O(\ell^{-2})$ ) the regions of phase space:

$$\begin{aligned}
(8.8) \quad & \langle F_1(|x| \leq C) F(|A-M| \geq K_0) \rangle_u^{\text{Heis}} \\
& \langle F(|A-M| \leq K_0) F_M \rangle_u^{\text{Heis}} \\
& \langle F_x \bar{F}_p F(|A-M| \geq K_0) \rangle_u^{\text{Heis}}.
\end{aligned}$$

To complete the proof of the main estimate with  $O(\ell^{-2})$  decay, up to logarithmic corrections in  $\ell$ , we need to bound the above three terms of the formula (8.8), which are referred below as terms I, II, III, with  $u \rightarrow Q^{1/2}u$ , by  $\ell^{-1}E(u)$ .

To this end, we estimate the scalar product as follows:

$$(8.9) \quad \left| \left\langle Q^{1/2}u, G_1 G_2 Q^{1/2}u \right\rangle \right| \leq C \|G_2 u\| \|G'_1 Q u\|,$$

for generic operators  $G$ , and with  $G'_1 \equiv G_1 + [Q^{1/2}, G_1]Q^{-1/2}$ .

Estimate of I:

$$(8.10) \quad \langle F_1(|x| \leq C) F(|A-M| \geq K_0) \rangle = \langle F_1 \rangle - \langle F_1 \bar{F} \rangle = -\langle F_1 \bar{F} \rangle.$$

Therefore, this last term is bounded by  $O(\ell^{-1} \ln \ell)E(u)$ , (with  $u \rightarrow Q^{1/2}u$ ), by applying Proposition 8.2.

Estimate of II:

$$(8.11) \quad \langle F F_M \rangle = \langle F F_M F_p \rangle + \langle F F_M \bar{F}_p F(|x| \leq C) \rangle + O(\ell^{-2}),$$

since, as we will show below, we only need to consider initial data with  $F(1/2\ell \leq Q \leq 2\ell)u = u$ , and the localization lemmas below, that imply  $\bar{F}_p F(|x| \geq C) F(Q \geq (1/2)\ell) = O(\ell^{-2})$ . The second term, on the right hand side of equation (8.11), is bounded by  $O(\ell^{-1} \ln \ell)E(u)$ , (with  $u \rightarrow Q^{1/2}u$ ), by applying Proposition 8.2 as before.

The first term, on the right hand side of equation (8.11), is bounded by  $O(\ell^{-1} \ln \ell)E(u)$ , (with  $u \rightarrow Q^{1/2}u$ ), by applying Proposition 8.1, since  $F_p Q F_p \geq c\ell F_p$ .

Estimate of III:

$$\langle F_x(|x| \geq C) F(|p| \leq \delta\ell) \rangle = O(\ell^{-2}),$$

again, by the localization lemmas below.

**Lemma 8.4. Localization lemma** *Let  $H = -\partial_x^2 + \ell^2 V(x)$  be defined as before in this section. Furthermore, we normalize  $V(0) = 1$ . Then, for all  $n > 0$ ,*

$$(i) \quad F(H \geq (1/2)\ell^2)F(|x| \geq c)F(|p| \leq \delta\ell) = O(\ell^{-n}),$$

for all  $c$  large enough.  $\delta < 1/2$ .

$$(ii) \quad F(H \geq 2\ell^2)F(|p| \leq \delta\ell) = O(\ell^{-n}),$$

$$(iii) \quad F(H \leq (1/2)\ell^2)F(|x| \leq \delta) = O(\ell^{-n}),$$

*Proof.* The proof follows the method of proving the Localization Lemma of [Sig-Sof1,2]: i) Let us denote by  $g = F(|x| \geq c)F(|p| \leq \delta\ell)$ . Then, we define

$$(8.12) \quad \bar{H} \equiv g^* H g$$

We have:

$$\bar{H} = F_p F_x (p^2 + V(x)\ell^2) F_p F_x \leq \delta' \ell^2.$$

Therefore  $F(\bar{H} \geq (1/2)\ell^2) = 0$ . Then, with  $\tilde{g}g = g$ , and all positive integers  $k, k'$ ,

$$(8.13) \quad \begin{aligned} \tilde{g}^* F(H \geq (1/2)\ell^2) \tilde{g} &= \tilde{g}^* \{F(H \geq (1/2)\ell^2) - F(\bar{H} \geq (1/2)\ell^2)\} \tilde{g} \\ &= \tilde{g}^* \int \hat{F}(\lambda) e^{iH\lambda} \int_0^\lambda e^{-iHs} [H - \bar{H}] e^{is\bar{H}} d\lambda ds \tilde{g} \\ &= \tilde{g}^* \int \hat{F}(\lambda) \int_0^\lambda Ad_g^{(k)}(e^{iH(\lambda-s)} [H - \bar{H}] Ad_g^{(k')} [e^{is\bar{H}} d\lambda] ds \tilde{g}. \end{aligned}$$

Direct computation shows that  $[g, H], [g, \bar{H}] = O(\ell)$ . Therefore,

$$(8.14) \quad [g, e^{itH}] = ce^{itH} \int_0^t e^{-isH} O(\ell) e^{+isH} ds = O(t\ell).$$

By repeatedly commuting  $g$  through the above expression, and using the fact that  $[g, O(\ell)] = O(1)$ , it follows that the multicommutators in equation 8.13 are bounded by  $c_k t^k \ell^k$ , for some constants  $c_k$ , depending only on the sharpness of the functions defining  $g$ . Since  $H - \bar{H} = O(\ell^2)$ , direct estimate of the  $L^2$  norm of the rhs of equation 8.13 gives:

$$(8.15) \quad \begin{aligned} \|\tilde{g}^* F(H \geq (1/2)\ell^2) \tilde{g}\| &= \|\tilde{g}^* \{F(H \geq (1/2)\ell^2) - F(\bar{H} \geq (1/2)\ell^2)\} \tilde{g}\| \\ &\leq c_n \int |\lambda^n \hat{F}(\lambda)| d\lambda O(\ell^{2+n-1}). \end{aligned}$$

Finally, using the construction of the function  $F$ , we have that

$$(8.16) \quad \int |\lambda^n \hat{F}(\lambda)| d\lambda \leq c_n \ell^{-2n+2}.$$

Putting it all together, we establish the following improved local decay estimate, for large  $\ell$ :

**Theorem 8.5.** *For the Hamiltonian with the Schwarzschild potential  $\ell^2 V(x)$ , with  $V$  analytic repulsive, we have the following estimate:*

$$(8.17) \quad \int_0^T \|F(|x| \leq r_0)\ell u\|^2 \leq c \ln \ell E(u).$$

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